

Multi-Platform Access to Digital Content

Hemant K. Bhargava

Graduate School of Management

University of California Davis

hemantb@gmail.com

July 16, 2013

Abstract

Distribution and consumption of media, entertainment, software, and other forms of information goods has been transformed in the last two decades. Increasingly, consumers want to access content on multiple platforms. Content providers have adopted widely different strategies for providing such access, ranging from independent pricing on each platform to single price for access to all platforms. How should content providers manage, and price, multi-platform demand? I frame the multi-platform design problem in terms of a choice between pure bundling (one price gets both platforms), mixed bundling (price each platform separately, and offer discount for getting both) and partial bundling (one platform is sold separately and is bundled with the second). When one platform is considered superior by all customers, then offering multi-platform discounts helps the firm if higher-value customers have greater propensity for multi-platform access; the choice of bundling strategy depends on certain ratios of valuations and contingent valuations for the two platforms. When consumer valuations for the traditional and emerging platforms are mutually independent with demand profiles, then full mixed bundling is optimal when the demand profiles for the two platforms are relatively similar; otherwise the superior platform included access to the weaker one which is also priced separately. When platforms behave more like substitutes, then such partial bundling is less likely to be optimal.

Multi-Platform Access to Digital Content

1 Introduction

Distribution and consumption of media, entertainment, software, and other forms of information goods has been transformed in the last two decades. Print products are now distributed and consumed digitally. For instance, traditional readers of *The Economist* or *Wall Street Journal* value accessing these products online despite having the print version. Digital content such as *Netflix* streaming and *Kindle* ebooks, previously accessed on television sets and computers, is now consumed on smartphones or tablets. Software previously installed on enterprise data centers now runs over the Internet cloud (e.g., Adobe Creative Cloud). This transformation was initially spurred by availability of inexpensive desktop computing in the 1980s, and then accelerated with the emergence and widespread adoption of the Internet and World Wide Web in the 1990s and the last decade. More recently, the “anytime, anywhere” computing enabled by proliferation of smartphones and tablets has again disrupted established patterns for consuming information, entertainment, and computing products. Another transformation is around the corner in the form of wearable computers such as eyeglasses and wrist watches.

I study this phenomenon using the economic lens of product bundling. Two distinct component products 1 and 2 (here, *access* to the product via the two channels or two platforms) can either be sold and priced individually (*separate component sales* at prices p_1, p_2 respectively), only as a bundle of both products (*pure bundle*), or both individually and as a bundle (*mixed bundle*); the last also admits the special case of *partial bundling* where only one component is sold separately, in addition to the bundle. The bundle price is denoted with p_B and, as depicted in Fig. 1, and the relation between the three prices reflects the selling strategy in use. There are several mechanisms by which bundling can raise producer profits: lowering the dispersion in consumers’ valuations (Stigler 1963; Adams

and Yellen 1976; Schmalensee 1984; McAfee et al. 1989), providing supply-side economies of scope (Evans and Salinger 2005; Surowiecki 2010), lowering consumer transaction costs or creating other demand-side conveniences and network effects (Lewbel 1985; Prasad et al. 2010), and creating strategic leverage (Burstein 1960; Carbajo et al. 1990; Eisenmann et al. 2011).

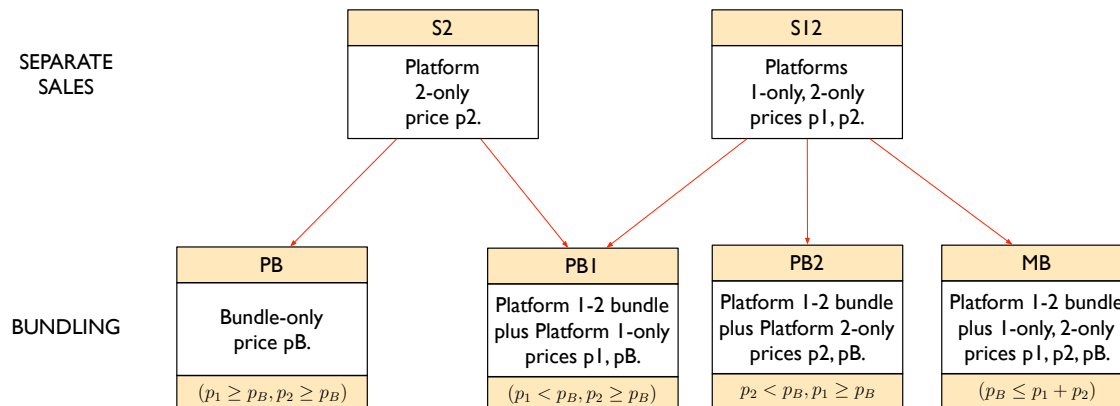


Figure 1: Strategies for providing access on multiple platforms. Platform 1 is the emerging platform while 2 is the traditional platform.

Many content providers have enabled multi-platform (or multi-channel)¹ access to content recently, but have employed widely different bundling strategies. For instance, *The New York Times* offers a *pure bundle* with free and full access to the online edition rolled into its print subscription. *The Economist* employs *mixed bundling*: users can subscribe to either just print or just digital, or to a discounted bundle of print and digital access. The *Wall Street Journal* employs a *partial mixed bundling*—or *tying*—strategy by pricing a digital-only edition and a digital+print bundle, but with no separate price for a digital-only product; thus, the print version is tied to the digital version (the tying good). Indeed, *The Economist* employed this approach until December 2012, before adopting mixed bundling (Ives 2012). Software vendors initially attempted to charge separate prices for each access platform, but many are now moving towards an integrated pure bundle price for multi-platform access to

¹To keep the exposition simple, I will use platform and channel interchangeably, and use the label multi-platform to refer to both multi-channel and multi-platform.

software, especially with the advent of cloud-based computing. Content aggregation and distribution firms such as *Comcast* limit subscriber access to the traditional consumption platform (television), and provide little or no formal content on the Internet platform. In contrast, *Netflix* allows subscribers to access their account from any supporting platform at no additional cost (a pure bundle). The premium content provider *HBO* also gives its TV subscribers free additional access on tablets and other platforms (under the *HBO GO* label) at no additional cost; *HBO GO* is not sold separately (implying that the firm sacrifices selling opportunities to the tens of millions of consumers who do not buy the cable or satellite package), though this possibility is being actively considered by the company (Wallenstein 2013).

This proliferation of selling strategies motivates the fundamental question analyzed in this paper: *how should content providers manage, and price, multi-platform demand?* The alternate strategies are labeled $S2, S12$ (for separate sales under just platform 2, and both platforms, respectively) and $PB, PB1, PB2, MB$ (representing pure bundling with no separate sales, partial bundling with separate access only on one platform, and full mixed bundling). The corresponding information sets (specifically, information about consumer demand preferences relating to single and multi-platform access) needed to implement each of these strategies are labeled $I_{S2}, I_{S12}, I_{PB}, I_{PB1}, I_{PB2}$, and I_{MB} respectively. I develop a model to analyze alternative strategies under a spectrum of market characteristics and consumer preferences for the multiple access platforms and platforms, and I study the nature of information required to implement the more “advanced” strategies involving the use of bundling. I describe how the choice of selling strategy is influenced by alternative relationships between a consumer’s valuation of the product under the traditional platform vs. their valuation under the new or emerging platform. A key tension studied in the model is whether and when, in their desire to fulfill existing customers’ need for add-on access, content providers should sacrifice the new and emerging segment of consumers for whom the new platform may in fact be the dominant or preferred one. This question is salient because

many content providers have responded to multi-platform demand by simply extending access to their traditional customers under a pure bundling framework in which purchasing the traditional good is a prerequisite to gain access under the new platform.

While multi-platform access is relevant to a wide spectrum of information goods as mentioned above, the present paper and model is most suited to digital *content* goods (such as newspapers and magazines, movies, music, video games) rather than to *computing* goods such as software. This is because the model does not presently account for repeat consumption over time (software, unlike a movie, is not consumed in a single session, and the same code may be executed hundreds of times) or the impact of direct network effects (which are present and strong for many software goods). These would be good extensions, but are best left out in a first-round model. The model is also less informative for products (such as TV shows) that are distributed through intermediaries who aggregate the product into a bundle of products from other competing producers. For such products, the economic distortions caused by cross-producer aggregation must be taken into account (see e.g., Bhargava 2012); again, these are not considered in the present model but would be a good future extension.

2 Model

The examples given in §1 span the space of bundling strategies—pure bundling, mixed bundling, partial bundling, and unbundled sales—validating the application of the bundling framework to the multi-platform access design question. To introduce the notation formally, let p_1, p_2 be the prices for purchasing platform i -only access (if offered), and let p_B (if offered, and if $p_B \leq p_1 + p_2$) be the price for bundled access. The firm employs a bundling strategy when it offers a positive bundle discount $\delta = p_1 + p_2 - p_B > 0$. It will be useful to describe the firm's strategy in terms of p_1, p_2 and δ (rather than p_B).

2.1 Consumer Valuations and Purchase Decisions

For consumer x , let $v_i(x)$ ($i=1, 2$) denote her standalone valuation for accessing the content via platform i -only, and let $v_B(x)$ be the valuation for the bundle of both platforms. For parsimony in notation we will employ the (x) qualifier only when required for clarity. The valuations v_i have continuous density f_i on the interval $[0, a_i]$, and cumulative distribution F_i (so that standalone demand for platform i at price p is $D_i(p) = 1 - F_i(p)$), which is common knowledge to consumers and the firm. Both demand functions have non-decreasing elasticity, i.e., the ratio $\frac{p f_i(p)}{1 - F_i(p)}$ is increasing in p . Without loss of generality, arrange consumers such that $v_2(x)$ is weakly increasing in x (i.e., $v_2'(x) \geq 0$), and that platform 2 is the one with superior demand profile, $D_2(p) \geq D_1(p) \forall p$ (with $a_2 \geq a_1$).² Generally, therefore, platform 2 would correspond to the *traditional* platform (usually, print or television or computer) because it tends to have the higher range of consumer valuations relative to the emerging platform (e.g., smartphone). Sometimes, the reverse might be true; for instance, *Cosmopolitan* magazine recently set a higher price for its Tablet edition than for a print subscription (Hagey 2013). The model is equally applicable to such reverse cases, and the reader can interpret platform 2 simply as being the one with a higher range of valuations.

In general, consumers will view multiple platforms as partial substitutes for accessing the same content, albeit with different user interface features or other details. Then, v_B will be below $v_1 + v_2$. Let $\tilde{v}_i \leq v_i$ be a consumer's *contingent* valuation for platform i , which is the valuation contingent on having platform j . The gap $\tilde{\phi} = (v_1 + v_2 - v_B) = (v_i - \tilde{v}_i) = (v_j - \tilde{v}_j)$ is a *bundle penalty* faced by the consumer. Its magnitude depends on product features, usage characteristics, or consumer preferences. For instance, the bundle penalty for multi-platform access to music may be low because it enables access while traveling, taking a walk, or sitting in an office. An e-book novel may present a high bundle penalty because it is generally read only once.

²Despite this ordering on the aggregate demand profile, a *specific* consumer might still have higher value under platform 1.

When $v_B < v_1 + v_2$, then consumers' attitudes towards multi-platform access can be summarized via two related concepts: the bundle penalty and, conversely, the increase in valuation from moving from single-platform to multi-platform access. For convenience and because platform 2 is designated as the traditional platform, we define both these concepts as proportional to the base valuation for platform 2.

Definition 1 (Consumer attitude towards multi-platform access). *For consumer x ,*

- *Penalty from overlap in platforms is $\tilde{\phi} = \frac{\phi}{v_2} = 1 - \frac{\tilde{v}_2}{v_2}$.*
- *Propensity for multi-platform access is $\psi = \frac{v_B - v_2}{v_2} = \frac{\tilde{v}_1}{v_2}$.*

With bundle penalty $\tilde{\phi} > 0$, and if the firm prices each platform separately, then some consumers who would otherwise have purchased both platforms at prices p_i will renege from one of these purchases, and consumers for whom $v_1 < p_1$ and $v_2 < p_2$ (but $v_1 + v_2 > p_1 + p_2$) will drop both platforms. If, however, the firm offers a bundle discount $\delta = p_1 + p_2 - p_B$, then consumers for whom $\tilde{\phi} \geq \delta$ would revert to buying both platforms. Let \mathcal{N}_i ($i=1, 2$) be the number of consumers who purchase platform i -only, and \mathcal{N}_B be the ones who buy both. Then, under the price vector (p_1, p_2, δ) ,

$$\mathcal{N}_{i=\text{platform } i\text{-only}} \equiv (v_i - p_i) \geq \max\{(v_i - p_i) + (\delta - \tilde{\phi}) + (v_j - p_j), (v_j - p_j), 0\} \quad (1)$$

$$\mathcal{N}_B = \text{multi-platform} \equiv (v_1 - p_1 + \delta - \tilde{\phi} + v_2 - p_2) \geq \max\{v_1 - p_1, v_2 - p_2, 0\} \quad (2)$$

For a more specific formulation of these functions, consider the simple case where $\tilde{\phi} = 0$ for all x , hence $v_B = v_1 + v_2$. Consumers' joint valuations for platforms 1 and 2 are distributed over a rectangle with sides of length a_1 and a_2 respectively. At any triplet of prices p_1, p_2, p_B (such that $p_i \leq \min\{a_i, p_B\}$ and $p_B \leq p_1 + p_2$) the market splits into 4 segments representing consumers who buy nothing, access under only one platform (1 or 2), or both platforms. Fig. 2 depicts the 4 regions, under three scenarios which exhaustively cover the order of p_B, a_1 and a_2 : $p_B \leq a_1 \leq a_2$; $a_1 \leq p_B \leq a_2$; and $a_1 \leq a_2 \leq p_B$. If $g(v_1, v_2)$ is the density of

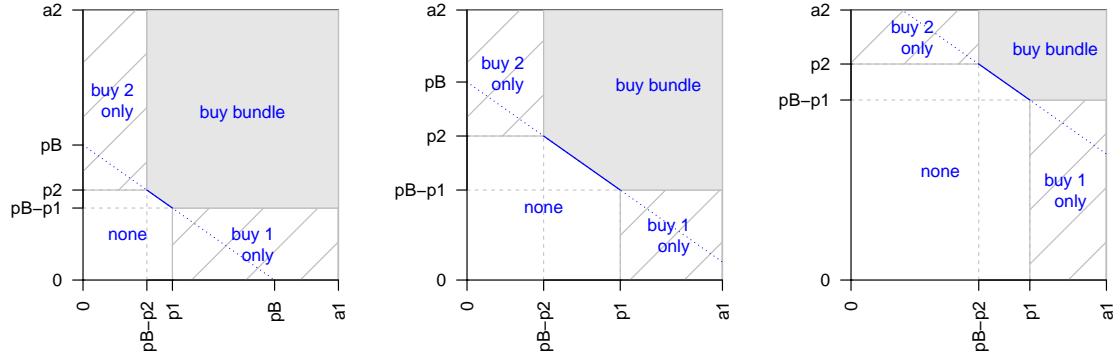


Figure 2: Product sales under prices p_1, p_2, p_B . The three panels cover the cases $p_B \leq a_1 \leq a_2$; $a_1 \leq p_B \leq a_2$; and $a_1 \leq a_2 \leq p_B$ respectively. Each point in the rectangle represents a consumer's valuations for the pair of platforms.

consumers with valuations v_i under platform i , then Eq. 1–2 reduce to

$$\mathcal{N}_1 = \int_0^{p_B - p_1} \int_{p_1}^{a_1} g(x, y) \mathbf{d}x \mathbf{d}y \quad (3a)$$

$$\mathcal{N}_2 = \int_{p_2}^{a_2} \int_0^{p_B - p_2} g(x, y) \mathbf{d}x \mathbf{d}y \quad (3b)$$

$$\mathcal{N}_B = \int_{p_B - p_1}^{a_2} \int_{p_B - p_2}^{a_1} g(x, y) \mathbf{d}x \mathbf{d}y - \int_{p_B - p_2}^{p_1} \left(\int_{p_B - p_1}^{p_B - x} g(x, y) \mathbf{d}y \right) \mathbf{d}x. \quad (3c)$$

2.2 Marginal and Joint Distribution of Valuations

The formulation and results of bundling models vary according to the dependence between a consumer's valuations for the component goods, and how such dependencies are distributed across the consumer population. Variations on this dimension map the model onto different application scenarios for multi-platform access. I will examine carefully chosen points across the space of possible formulations, corresponding to zero, positive and negative correlation (see Fig. 3). While these designs are not exhaustive, systematic comparison and contrast across these designs will yield general insights regarding how the inter-relationship between demand preferences impacts the multi-platform access strategy offered by the producer.

Zero correlation is the neutral case where valuation under the traditional platform pro-

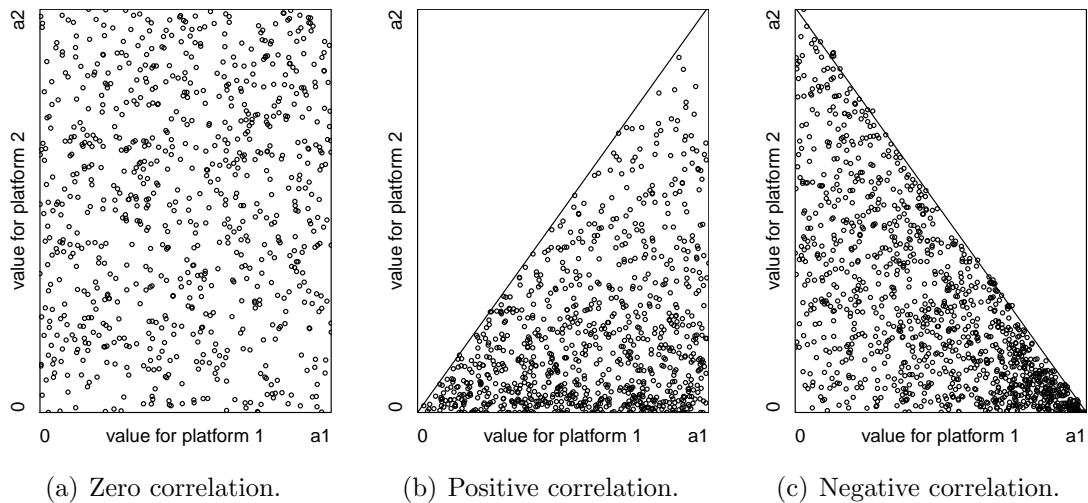


Figure 3: Sample distribution of product valuations for platforms 1-2 under zero, positive, and negative correlation. In each plot, each dot represents a pair of consumer valuations, for platform 1 (x-value) and platform 2 (y-value).

vides, probabilistically, no information regarding valuation under the emerging platform. Negative correlation reflects products for which the traditional user base is quite conservative with respect to changing technology or delivery mechanisms. It might apply when the traditional platform uniquely retains features that are essential for higher-end consumption. For instance, consider a software product with multiple complex menus and rich multimedia displays in multiple windows. A new smartphone interface for this product may attract previous non-users but would have little value to high-value traditional users. Positive correlation, in contrast, is likely when the traditional user base embraces the new or emerging platform or views it as an extension, e.g., for news reports and magazines, or for software products with relatively simple input-output displays. It might also describe specialized devices carried by personnel such as sales staff, delivery staff, or health care professionals. A smartphone or tablet interface provides additional opportunities to consume the content.

2.3 Framework for Analysis

The multi-platform strategy design problem can be posed as an optimization problem in the three prices p_1, p_2, p_B .

$$\begin{aligned} \underset{p_1, p_2, p_B}{\text{Maximize}} \quad & \Pi = p_1 \mathcal{N}_1 + p_2 \mathcal{N}_2 + p_B \mathcal{N}_B \\ \text{s.t.} \quad & 0 \leq p_1 \leq p_B, \quad 0 \leq p_2 \leq p_B, \quad p_B \leq p_1 + p_2. \end{aligned} \quad (4)$$

While detailed implementation of multi-platform strategy requires solving for optimal prices p_1, p_2, p_B , many of the insights regarding strategy design can be framed in terms of boundary conditions for this optimization problem. Specifically, when bundle penalty $\tilde{\phi} = 0$ for all consumers, mixed bundling is relevant exactly when the problem has an interior solution in which $p_B < p_1 + p_2$ (besides $p_1, p_2 < p_B$). Similarly, pure bundling is practiced when the optimal solution is at the boundary $p_1 = p_2 = p_B$, while the two partial bundling strategies occur when the optimal solution is at the respective boundary $p_i = p_B$ (with $p_j < p_B$). Finally, applications where bundling strategy is irrelevant (and separate selling is sufficient) map to conditions under which the optimal solution to the problem occurs at the boundary $p_B = p_1 + p_2$.

2.4 Benchmark: Single-Platform Strategy

For comparison purposes, pretend that the firm wishes to provide access only via a single platform, either 1 or 2. If platform i is offered at price p_i , then there exists a unique indifference point (marginal consumer) \hat{x}_i such that a fraction $(1 - F(\hat{x}_i))$ of the market purchases access. Using standard price optimization techniques, the optimal value of \hat{x}_i is where the demand elasticity equals 1 (assuming zero marginal cost). Hence, for the two

cases of selling either platform 1-only or platform 2-only, these indifference points are

$$\hat{x}_1 = \text{Sol.} \left[\frac{f(x)}{1 - F(x)} \times \frac{v_1(x)}{\frac{\partial}{\partial x}(v_1(x))} = 1 \right] \quad (5)$$

$$\hat{x}_2 = \text{Sol.} \left[\frac{f(x)}{1 - F(x)} \times \frac{v_2(x)}{\frac{\partial}{\partial x}(v_2(x))} = 1 \right]. \quad (6)$$

Because $p_i = v_i(x)$ for the marginal consumer, the firm's optimal profit in the two cases are, respectively, $(1 - F(\hat{x}_1))v_1(\hat{x}_1)$ and $(1 - F(\hat{x}_2))v_2(\hat{x}_2)$. The above terms can easily be extended to the case of constant positive marginal costs, simply by replacing the valuation v_i with the surplus from trade $v_i - c_i$.

A second useful benchmark is a restrictive case under vertically differentiated platforms (discussed in further detail in §3.1), wherein every consumer has higher valuation for platform 2 than for platform 1. The restriction is that while the firm covers both platforms, it restricts each consumer to purchasing access only on a single platform (or, equivalently, that bundle penalty is so large that no consumer wants multi-platform access). There are now two indifference points separating platform-2 buyers (highest x) from platform-1 buyers and those lowest- x customers who purchase nothing. From the vertical differentiation literature, the first indifference point \hat{x}_{12} (which separates platform-2 buyers from platform-1 buyers) is defined as

$$\hat{x}_{12} = \text{Sol.} \left[\frac{f(x)}{1 - F(x)} \times \frac{v_2(x) - v_1(x)}{\frac{\partial}{\partial x}(v_2(x) - v_1(x))} = 1 \right], \quad (7)$$

while the second (which separates platform-1 buyers from non-buyers) is exactly the \hat{x}_1 in Eq. 5. Positive sales occur for both platforms when $\hat{x}_{12} > \hat{x}_1$, for which Bhargava and Choudhary 2008 provide the condition

$$\frac{\partial}{\partial x} \left(\frac{v_1(x)}{v_2(x)} \right) \Big|_{\hat{x}_1} < 0, \quad (8)$$

When Eq. 8 fails, then the firm sells platform 2 only (with indifference point \hat{x}_2 in Eq. 6), because adding platform 1 to the mix fails to improve profit.

3 Multi-Platform Strategy

3.1 Positive Correlation: Vertically differentiated platforms

Suppose that all consumers consider one platform (platform 2) more attractive. This is often the case for software products which have rich multimedia displays and require complex manipulations. For instance, consider software applications from Adobe and Autodesk, both of which have strategically chosen to offer software as a service over the Internet. Most users of Autodesk's AutoCad or Adobe's Illustrator would prefer accessing the application via full-fledged computers over smartphone and tablet devices. Since multi-platform access provides higher valuation than any single platform, we have $v_B > v_2 > v_1$ for all consumers. Without loss of generality, assume x 's are ordered such that $v_1'(x) \geq 0$. Let X_i be the set of consumers who purchases access under platform i (either i alone or multi-platform), given prices p_1, p_2, p_B . Following the literature on vertical differentiation, assume the single-crossing property on net utility as specified below.

Assumption 1 (Single-Crossing Property). *With x 's arranged such that $v_1'(x) \geq 0$, the incremental benefit from higher-valued platforms is increasing in x : $\frac{\partial}{\partial x} (v_2(x) - v_1(x)) \geq 0$ and $\frac{\partial}{\partial x} (v_B(x) - v_i(x)) = \frac{\partial \tilde{v}_i}{\partial x} \geq 0$.*

The single-crossing property ensures that at any triplet of prices, the highest- x consumers (but possibly none) will purchase the bundle; the next-highest (again, possibly none) purchase platform 2 and then 1 in sequence; while the lowest- x consumers will purchase nothing. Of the different market share combinations implied by this, one can be trivially ruled out, that only platform 1 is sold; trivially, the firm could increase profit by setting prices such that the same group buys platform 2 instead. Standard incentive compatibility constraints (and

the individual rationality constraint for platform 1) yield the following indifference points,

$$v_B(\hat{x}_{B2}) - p_B = v_2(\hat{x}_{B2}) - p_2 \quad (9)$$

$$v_2(\hat{x}_{12}) - p_2 = v_1(\hat{x}_{12}) - p_1 \quad (10)$$

$$v_1(\hat{x}_1) - p_1 = 0. \quad (11)$$

If these points are strictly ordered with $0 < \hat{x}_1 < \hat{x}_{12} < \hat{x}_{B2}$, then they define a mixed bundling solution where $\mathcal{N}_B = (1 - F(\hat{x}_{B2}))$ (customers with $x \geq \hat{x}_{B2}$ purchase multi-platform access), $\mathcal{N}_2 = (F(\hat{x}_{B2}) - F(\hat{x}_{12}))$, and $\mathcal{N}_1 = (F(\hat{x}_{12}) - F(\hat{x}_1))$. From first-order conditions, the firm's profit,

$$\Pi = (1 - F(\hat{x}_{B2})) (v_B(\hat{x}_{B2}) - v_2(\hat{x}_{B2})) + (1 - F(\hat{x}_{12})) (v_2(\hat{x}_{12}) - v_1(\hat{x}_{12})) + (1 - F(\hat{x}_1)) v_1(\hat{x}_1)$$

is maximizing by setting \hat{x}_1 as in Eq. 5, \hat{x}_{12} as in Eq. 7 and \hat{x}_{B2} as below.

$$\hat{x}_{B2} = \text{Sol.} \left[\frac{f(x)}{1 - F(x)} \times \frac{v_B(x) - v_2(x)}{\frac{\partial}{\partial x} (v_B(x) - v_2(x))} = 1 \right] \quad (12)$$

If the constraint $\hat{x}_1 < \hat{x}_{12}$ is violated (no one buys platform 1 alone), then the partial bundling menu $\{2, B\}$ is plausible, with indifference points \hat{x}_2 as in Eq. 6 and \hat{x}_{B2} as in Eq. 12. If $\hat{x}_{12} < \hat{x}_{B2}$ is violated (no one buys platform 2 alone), then the menu $\{1, B\}$ is plausible, with indifference points \hat{x}_1 (Eq. 5) and \hat{x}_{B1} defined analogously to Eq. 12. When neither of the partial bundles yields strict ordering in the indifference points, then the pure bundle solution is characterized by the indifference point \hat{x}_B defined analogously to Eq. 5. Hence the alternative multi-platform selling strategies map to the interior vs. boundary optima to the firm's profit maximization problem.

Proposition 1. *Under Assumption 1 and non-decreasing demand elasticity,*

1. *full mixed bundling is optimal when Eq. 8 holds and, with respect to valuation v_2 , the ratio (value difference upon adding platform 1 vs. dropping down to platform 1) is increasing at \hat{x}_{12} ,*

$$\frac{\partial}{\partial x} \left(\frac{\tilde{v}_1(x)}{v_2(x) - v_1(x)} \right) \Big|_{\hat{x}_{12}} > 0 \quad \left(\text{i.e., } \frac{\tilde{v}_2(x)}{v_2(x) - v_1(x)}, \frac{\tilde{v}_1(x)}{\tilde{v}_2(x)} \text{ increasing at } \hat{x}_{12} \right). \quad (13)$$

2. *a partial bundle $\{1, B\}$ is better when Eq. 13 fails, but propensity for multi-platform access (relative to platform 1 only) is increasing at \hat{x}_1 ,*

$$\frac{\partial}{\partial x} \left(\frac{\tilde{v}_2(x)}{v_1(x)} \right) \Big|_{\hat{x}_1} > 0. \quad (14)$$

3. *a partial bundle $\{2, B\}$ is better when Eq. 8 fails but propensity for multi-platform access (relative to platform 2 only) is increasing at \hat{x}_2 ,*

$$\frac{\partial}{\partial x} \left(\frac{\tilde{v}_1(x)}{v_2(x)} \right) \Big|_{\hat{x}_2} > 0. \quad (15)$$

4. *pure bundling (single price for accessing both platforms) is optimal when Eqs. 14–15 fail, i.e., higher-value consumers have lower propensity for multi-platform access.*

While the above provides exact conditions for optimality of the full mixed bundling strategy (or, conversely, for $\mathcal{N}1$ or $\mathcal{N}2$ to be zero), it can also be applied to develop a more practical understanding of when $\mathcal{N}1$ and/or $\mathcal{N}2$ become too small to be viable. In the multi-platform strategy, it is worthless to sell platform 1 by itself when the ratio $\frac{v_1(x)}{v_2(x)}$ is quite similar (or increasing) for all x in the neighborhood of \hat{x}_1 . That is, not only should the decreasing ratio condition be satisfied, but in fact the ratio should be sharply decreasing in order for the strategy to be truly impactful. Incidentally, the condition is identical to that needed to ensure positive sales of platform 1 in the vertical differentiation benchmark with the restriction that each consumer purchases at most one platform. Similarly, separate sales of platform 2 are meaningless when the ratio $\frac{\tilde{v}(x)}{v_2(x) - v_1(x)}$ is quite similar (or decreasing) for all x in the neighborhood of \hat{x}_{12} . Notably, these pivotal points \hat{x}_1 and \hat{x}_{12} are simply the values that are featured in a standard two-product vertical differentiation pricing problem.

Proposition 1 generalizes Banciu et al. 2010’s result that was derived for the restricted case of constant marginal valuations for quality ($v_2 = \lambda v_1$ with $\lambda > 1$, and $v_B = \gamma v_1$ with $\gamma > \lambda$), implying constant rather than strictly increasing ratio $\frac{\partial \tilde{v}_i(x)}{\partial v_j(x)}$. Then there is no benefit from offering vertically differentiated products, and selling only the highest quality (i.e., the pure bundle) is optimal and yields identical allocation and profits as separate selling (because $X_1 = X_2$). However, when higher-valuation customers “care sufficiently more” for multi-platform access (i.e., they have a higher proportional increase in value relative to single-platform access), then the firm can profitably segment the market. In this case, there are two choices because the single-platform segment can be given either platform 1 or 2. Because platform 2 is deemed the higher-valuation platform, offering it in a partial bundling strategy would cause more cannibalization and competition with the multi-platform bundle. Hence the optimal partial bundle has the lower-valuation platform 1, while those who seek platform 2 are convinced to pay a slightly higher price in return for the 1-2 bundle.

It is hard to apply Proposition 1 definitively to specific products without accurate empirical data about users’ demand preferences. However, as an example of its relevance, consider the following conjecture with regard to Autodesk’s products for 3D design in engineering and entertainment. Autodesk products were traditionally available for computer workstations but now are also available for mobile devices. These products are used by both industry professionals and mass consumers such as amateurs, students, and hobbyists. Not only do professionals have higher value for computer-based access, but likely obtain only a small proportional increase in valuation upon adding smartphone or tablet access which provides a relatively crippled user experience for such complex applications. The reverse might be true for mass users who can accomplish a lot with a tablet device, relative to their value for workstation use. Hence (with the workstation as the base platform) these professionals have lower relative propensity for multi-platform access, and the partial bundle $\{2, B\}$ may not be profitable. However, the property is reversed when considering the mobile device as the base platform ($j=1$ in Proposition 1). The ratio $\frac{\tilde{v}_i(x)}{v_j(x)}$ is now increasing in x , making partial

bundling optimal. The firm should offer a relatively low-priced access over mobile devices, and bundle mobile access for free for buyers of a premium-priced workstation product. Of course, in practice, it might also be useful to offer a limited-feature workstation version to mass consumers at a lower price.

Another notable aspect about Proposition 1 is that it provides guidance about the design of a multi-platform strategy without requiring much additional information than what the firm might need to sell the platforms separately without a coordinated bundle discount. Applying Proposition 1 requires knowledge of platform valuations v and contingent valuations \tilde{v} , especially around the critical points $\hat{x}_1, \hat{x}_2, \hat{x}_{12}$. Of these, knowledge about valuations at \hat{x}_1 and \hat{x}_2 is needed simply in order to implement a single-platform strategy with either platform 1 or 2 respectively. And knowledge about contingent valuations is needed simply to price the two platforms if the firm were selling both even without a bundle discount. The only additional information needed is whether the various ratios in the proposition are increasing or decreasing in x especially at these critical points. It is also notable that the result holds regardless of the specific distribution of valuations as long as the demand functions have non-decreasing demand elasticities; this requirement is satisfied by distributions that have a monotone hazard rate. The result is also applicable under various levels and forms of bundle penalty, and does not require specific functional relationships between valuations of each platform.

3.2 Independent Platforms

Consider the case where the two platforms are essentially independent in value-provision in the sense that i) each consumer's value from one platform is independent of her having or not having the second platform (i.e., $v_B = v_1 + v_2$, or bundle penalty $\tilde{\phi} = 0$), and ii) a consumer's valuation for one platform provides no additional information about her value for the second platform (the distributions f_1 and f_2 have zero correlation). As noted earlier, the two platforms may be asymmetric in terms of total market demand, with platform 2

(labeled as the existing technology) having the superior demand profile. For mathematical and computational convenience in analyzing this case, it is useful to employ a specific distribution of standalone valuations for each platform. In absence of any other information, we assume that these marginal distributions f_1 and f_2 have uniform density.

Assumption 2 (Demand for each platform). *Consumer valuations for obtaining service via platform 1 (or 2) are distributed uniformly in an interval $[0, a_1]$ (respectively, $[0, a_2]$).*

Assumption 3 (Zero Correlation). *A consumer's valuations for platforms i and j are independent. The joint distribution covers the unit rectangle $[(0, 0), (0, a_2), (a_1, a_2), (a_1, 0)]$.*

Assumption 4 (Zero Bundle Penalty). *A consumer's multi-platform valuation v_B is the sum of her valuations for each platform alone.*

Building on the results of Bhargava 2013 yields the following insights. Let's start with the simpler case of pure digital goods for which $w_1=0=w_2$ is a reasonable assumption. From Proposition 3 in Bhargava 2013, full mixed bundling is optimal when the two platforms have relatively similar demand profiles (specifically, $a_1 \in [\frac{a_2}{2}, a_2]$), while partial bundling with platform 1 and the bundle becomes dominant otherwise (when $a_1 \leq \frac{a_2}{2}$). The bundle discount—computed as a proportion of the sum of the costs of each single platform, $\frac{p_1+p_2-p_B}{p_1+p_2}$ —also depends on the relative similarity of demand profiles, i.e., on the attractiveness of mixed bundling. It is easily seen that the proportional bundle discount increases with a_1 ; this is because demand profiles for the two platforms become more balanced as a_1 increases towards a_2 . Conversely, bundle discount decreases with a_2 because the profiles get sharply distinct—leading to a partial bundling optima—as a_2 increases and becomes further distant from a_1 .

Proposition 2. *When consumer valuations for the traditional and emerging platforms are mutually independent with demand profiles satisfying Assumptions 2–4, then the optimal strategy is to*

1. *sell separate access to each platform as well as discounted access to the bundle, when the demand profiles for the two platforms are relatively similar (i.e., a_1 is no less than approximately $\frac{1}{2} \frac{a_2 - w_2}{a_2}$),*
2. *provide buyers of the traditional platform with free access to the emerging platform but also offer for-fee access to the emerging platform only, when the demand profile for the emerging platform is sufficiently weak (a_1 is less than approximately $\frac{1}{2} \frac{a_2 - w_2}{a_2}$).*

The bundle discount is highest when per-unit costs are zero, decreases as unit costs increase, and is most sensitive to the unit cost of providing access via the emerging platform.

The core insight from Proposition 2 is relatively easy to state. Practice mixed bundling when the two platforms are somewhat balanced in demand profiles. When they are not, then offer the weak-demand platform separately (targeting those consumers who do value it highly), but also bundle it for free with the strong-demand one. While this latter strategy resembles “tying” it does not have the typical motive behind tying, that is to leverage a strong product into driving sales of the weaker product or one that has strong competition. Instead the main reason to avoid selling platform 2 by itself is simply that a product line with both platform 2-alone and the bundle would have too much internal cannibalization because most consumers would have relatively similar values for these two items when platform 1 valuations are very weak.

The condition of independent and additive valuations applies to, or approximates, several applications of multi-platform access. For instance, consider two platforms—workstation and tablet—for an enterprise information system that lets users view informational reports as well as manipulate the data in the system. Workstation access to the system might be needed for making full use of multimedia and graphical report formats or for entering large amounts of data. Tablet device access might provide mobile access and be highly convenient for browsing predefined reports and minimal data entry. The tablet platform provides *added* functionality (mobility, convenience) rather than substitute the value provided by workstation access. The

zero correlation assumption may also fit this setting because users' valuation for workstation provides little information about their value for mobility and convenience. Proposition 2 then provides clear guidance for designing the multi-platform strategy. Exact prices for each platform and the bundle can be determined using formulæ given in Bhargava 2013. Moreover, prior work by Long 1984; McAfee et al. 1989 suggests that the design guidelines provided in Proposition 2 are actually robust and applicable even when valuations are sub-additive and when the distribution of consumer valuations are not independent (as long as they are not too positively correlated).

3.3 Constant Bundle Penalty

Assumption 5 (Constant bundle penalty). *All consumers perceive the same bundle penalty $\tilde{\phi} = v_1 + v_2 - v_B \geq 0$.*

Next, consider the case where the bundle penalty $\tilde{\phi}$ is identical for all consumers. The constant penalty assumption has previously been used in Armstrong 2013, while Venkatesh and Kamakura 2003 employed a constant *proportional* penalty equaling $\theta(v_1 + v_2)$. Now, because of the constant value $\tilde{\phi}$ in Eq. 1-2 the multi-platform strategy analysis can be reduced to an equivalent mathematical problem by setting bundle penalty to 0 (i.e., $v_B = v_1 + v_2$) but adding $\tilde{\phi}$ to the effective bundle price viewed by consumers who are considering multi-platform consumption. That is, for now let \tilde{p}_B represent the actual bundle price set by the firm, so that the p_B in Fig. 2 and Eq. 3 (which define the sales levels $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_B$) actually refers to $p_B = \tilde{p}_B + \tilde{\phi}$, while the bundle price in the firm's profit function (Eq. 4) is replaced by $p_B - \tilde{\phi}$. Further adopting Assumptions 2 and 3 (uniform distribution of valuations for each platform, and zero correlation between the two distributions), the optimization problem is

refined to

$$\begin{aligned} \text{Maximize}_{p_1, p_2, p_B} \quad & \Pi = p_1(a_1 - p_1)(p_B - p_1) + p_2(a_2 - p_2)(p_B - p_2) \\ & + (p_B - \tilde{\phi}) \left((a_2 - p_B + p_1)(a_1 - p_B + p_2) - \frac{(p_1 + p_2 - p_B)^2}{2} \right) \\ \text{s.t.} \quad & 0 \leq p_1 \leq \min\{a_1, p_B - \tilde{\phi}\}, \quad 0 \leq p_2 \leq \min\{a_2, p_B - \tilde{\phi}\}, \quad p_B \leq p_1 + p_2 + \tilde{\phi}. \end{aligned} \quad (16)$$

First-order conditions with respect to p_1 and p_2 are quadratic, implying the possibility (combined with the third condition) that the optimization problem has at least 4 critical points. The following result prunes the list of candidate optima.

Lemma 1 (Component Prices). *In the optimal mixed bundle solution, the component prices p_1 and p_2 are uniquely specified as a function of the bundle price p_B ,*

$$p_1 = \frac{1}{6} \left(3p_B + (2a_1 - \tilde{\phi}) - \left| \sqrt{(3p_B - 2a_1 + \tilde{\phi})^2 + 12\tilde{\phi}(a_1 - p_B)} \right| \right) \quad (17a)$$

$$p_2 = \frac{1}{6} \left(3p_B + (2a_2 - \tilde{\phi}) - \left| \sqrt{(3p_B - 2a_2 + \tilde{\phi})^2 + 12\tilde{\phi}(a_2 - p_B)} \right| \right). \quad (17b)$$

Lemma 1 facilitates insights into the nature of the optimal solution—whether it involves full, partial, or no bundling, and under what conditions. Bhargava 2013 demonstrated (under the special case $\tilde{\phi}=0$) that full bundling is optimal when the range of valuations under the two platforms is comparable, and that partial bundling (with only platform 1 being priced independently) is optimal when the range is substantially different. Extending the result to the $\tilde{\phi}>0$ case, we find that

Proposition 3. *The optimal solution to the multi-platform access design problem (Eq. 16) is*

1. *a partial bundle strategy—sell the bundle and sell access on the new platform 1 alone—when $a_2 \geq 2a_1 + \tilde{\phi}$. Optimal prices are $p_B = a_2$ and $p_1 = \frac{1}{6} \left((3a_2 + 2a_1 - \tilde{\phi}) - \sqrt{(3a_2 - 2a_1 - \tilde{\phi})^2 + 4a_1\tilde{\phi}} \right)$*
2. *sell both platforms separately and as a bundle (mixed bundling) when $a_2 < 2a_1 + \tilde{\phi}$.*

A full mixed bundling strategy is optimal when the two platforms have comparable aggregate distribution of valuations; otherwise, a partial bundling strategy is optimal wherein the platform which commands far stronger valuations is not offered separately. The partial bundle is consistent with the maximal differentiation principle: the firm's product line contains the two options that are most different from each other, avoiding internal competition between two options that have relatively similar value. Compare the solution with the special case where the two platforms have no substitution effect (i.e., $\tilde{\phi}=0$, no bundle penalty). Then partial bundling becomes active when a_2 exceeds $2a_1$. Hence the insight from Proposition 3 is that a positive bundle penalty makes partial bundling *less* likely; another way to interpret the result is that a greater difference in valuations is needed before switching from mixed bundling to the partial bundling strategy wherein the elimination of separate sales for platform 2 makes bundling more dominant.

4 Conclusion

This paper is still a work in progress. The core question of designing a multi-platform strategy is framed in terms of bundling two related platforms, creating a choice between pure bundling (one price gets both platforms), mixed bundling (price each platform separately, and offer discount for getting both) and partial bundling (one platform is sold separately and is bundled with the second). Mathematical models for bundling are inherently complex to analyze, and the relevant model for this paper has multiple complexities: correlated demand distributions; sub-additive valuations; and asymmetric platforms. Nevertheless, I have been able to develop some useful insights regarding the design of the multi-platform strategy. When the platforms are vertically differentiated, then offering multi-platform discounts helps the firm if higher-value customers have greater propensity for multi-platform access. When consumer valuations for the traditional and emerging platforms are mutually independent with demand profiles, then full mixed bundling is optimal when the demand profiles for the

two platforms are relatively similar; otherwise the superior platform included access to the weaker one which is also priced separately. When platforms behave more like substitutes, then such partial bundling is less likely to be optimal. While these are preliminary results, I believe that the model presented here lays a useful foundation for further exploration and analysis, and intend to continue this work in the next few days up to the Symposium and later.

References

- Adams, William James and Janet L. Yellen (1976). “Commodity Bundling and the Burden of Monopoly”. In: *The Quarterly Journal of Economics* 90.3, pp. 475–498. URL: <http://ideas.repec.org/a/tpr/qjecon/v90y1976i3p475-98.html>.
- Armstrong, Mark (2013). “A more general theory of commodity bundling”. In: *Journal of Economic Theory* 148, pp. 448–472.
- Banciu, Mihai et al. (2010). “Bundling Strategies When Products Are Vertically Differentiated and Capacities Are Limited”. In: *Management Science* 56.12, pp. 2207–2223.
- Bhargava, Hemant K. (2012). “Retailer-Driven Product Bundling in a Distribution Channel”. In: *Marketing Science* 31.6, pp. 1014–1021.
- (2013). “Mixed Bundling of Two Independently-Valued Goods”. In: *Management Science*.
- Bhargava, Hemant K. and Vidyanand Choudhary (2008). “When is Versioning Optimal for Information Goods?” In: *Management Science* 54 (May 2008), pp. 1029–1035.
- Burstein, M. L. (1960). “The Economics of Tie-In Sales”. In: *The Review of Economics and Statistics* 42 (1 1960), pp. 68–73.
- Carbajo, José et al. (1990). “A Strategic Motivation for Commodity Bundling”. In: *The Journal of Industrial Economics* 38.3.
- Eisenmann, Thomas et al. (2011). “Platform envelopment”. In: *Strategic Management Journal* 32.12, pp. 1270–1285. DOI: [10.1002/smj.935](https://doi.org/10.1002/smj.935).

- Evans, David S. and Michael A. Salinger (2005). “Why Do Firms Bundle and Tie? Evidence from Competitive Markets and Implications for Tying Law”. In: *Yale Journal on Regulation* 22.1.
- Hagey, Keach (2013). “Magazines Cross The Digital Divide”. In: *The Wall Street Journal* January 19, B1. URL: <http://online.wsj.com/article/SB10001424127887323706704578227880541302630.html>.
- Ives, Nat (2012). “Digital Subs Rising, The Economist Unbundles Tablet Editions From Print”. In: *Ad Age* November 30. URL: <http://adage.com/article/media/economist-unbundles-tablet-editions-print-subs/238566/>.
- Lewbel, Arthur (1985). “Bundling of substitutes or complements”. In: *International Journal of Industrial Organization* 3.1, pp. 101–107.
- Long, J. (1984). “Comments on “Gaussian Demand and Commodity Bundling””. In: *Journal of Business* 57.1, S235–S246.
- McAfee, R. Preston et al. (1989). “Multiproduct Monopoly, Commodity Bundling, and Correlation of Values”. In: *The Quarterly Journal of Economics* 104.2, pp. 371–83.
- Prasad, Ashutosh et al. (2010). “Optimal Bundling of Technological Products with Network Externality”. In: *Management Science* 56.12, pp. 2224–2236.
- Schmalensee, Richard (1984). “Gaussian Demand and Commodity Bundling”. In: *The Journal of Business* 57.1, S211–230.
- Stigler, George (1963). “United States v. Loew’s Inc: A Note on Block-Booking”. In: *The Supreme Court Review*, pp. 152–157.
- Surowiecki, James (2010). “Bundles of Cable”. In: *The New Yorker* (25 2010).
- Venkatesh, R. and Wagner Kamakura (2003). “Optimal Bundling and Pricing under a Monopoly: Contrasting Complements and Substitutes from Independently Valued Products”. In: *Journal of Business* 76.2, pp. 211–232. URL: <http://econpapers.repec.org/RePEc:ucp:jnlbus:v:76:y:2003:i:2:p:211-232>.

Wallenstein, Andrew (2013). “Time to Free HBO Go From Its TV Shackles”. In: *Variety* March 26. URL: <http://variety.com/2013/digital/news/time-to-free-hbo-go-from-its-tv-shackles-1200329375/>.

A Appendix

Proof of Proposition 1. First we derive conditions for pure bundling, i.e., offering *only* the firm’s “highest quality” product which is multi-platform access. Applying Lemma 3 in Bhargava and Choudhary 2008, selling bundled multi-platform access only (i.e., only selling the most superior product quality) is optimal when it is superior to each partial bundling strategy, $\{1, B\}$ and $\{2, B\}$. From Proposition 2 in the same paper, these conditions are, respectively, $\frac{\partial}{\partial x} \left(\frac{v_1(x)}{v_B(x)} \right) \geq 0$ and $\frac{\partial}{\partial x} \left(\frac{v_2(x)}{v_B(x)} \right) \geq 0$. Inverting the ratios, the conditions are $\frac{\partial}{\partial x} \left(\frac{v_B(x)}{v_1(x)} \right) \leq 0$ and $\frac{\partial}{\partial x} \left(\frac{v_B(x)}{v_2(x)} \right) \leq 0$. Finally, writing v_B as $v_i + \tilde{v}_j$ yields the conditions as $\frac{\partial}{\partial x} \left(\frac{\tilde{v}_2(x)}{v_1(x)} \right) \leq 0$ and $\frac{\partial}{\partial x} \left(\frac{\tilde{v}_2(x)}{v_2(x)} \right) \leq 0$ respectively. The converses of these conditions yield the cases where partial bundling beats pure bundling.

Next, consider the conditions for full mixed bundling to be optimal. Now, because of non-decreasing demand elasticities, the LHS terms are weakly increasing and thus each have a unique intersection with the constant horizontal line at 1. Hence the ordering constraint $0 < \hat{x}_1 < \hat{x}_{12} < \hat{x}_{B2}$ is satisfied if the LHS term (after dropping the identical component $\frac{f(x)}{1-F(x)}$) in the first equation is lower than that in the second, at \hat{x}_{12} ; and if the second term is lower at \hat{x}_1 than the third. Consider the second requirement, i.e., that $\frac{\partial}{\partial x} \left(\frac{v_2(x)-v_1(x)}{v_2(x)-v_1(x)} \right) \leq \frac{\partial}{\partial x} \left(\frac{v_1(x)}{v_2(x)} \right)$. This is identical to $v_2(x) \frac{\partial v_1(x)}{\partial x} \leq v_1(x) \frac{\partial v_2(x)}{\partial x}$, which in turn is identical to the condition $\frac{\partial}{\partial x} \left(\frac{v_1(x)}{v_2(x)} \right) \leq 0$ at \hat{x}_1 . Similarly, the first requirement is $\frac{\partial}{\partial x} \left(\frac{v_2(x)-v_1(x)}{\tilde{v}_1(x)} \right) \leq 0$, equivalent to $\frac{\partial}{\partial x} \left(\frac{\tilde{v}_2(x)-\tilde{v}_1(x)}{\tilde{v}_1(x)} \right) \leq 0$ and yields $\frac{\partial}{\partial x} \left(\frac{\tilde{v}_1(x)}{\tilde{v}_2(x)} \right) \geq 0$ at \hat{x}_2 . The format of the conditions in the proposition follows from the equivalence between the sign of the derivatives of the following terms: $\frac{\tilde{v}_1(x)}{\tilde{v}_2(x)}$; $\frac{\tilde{v}_1(x)}{v_2(x)-v_1(x)}$; $\frac{\tilde{v}_2(x)}{v_2(x)-v_1(x)}$. ■

Proof of Proposition 2. To compute bundle discount $\delta = p_1 + p_2 - p_B$, use Eq 5a-b from

Bhargava 2013, and rearrange,

$$\delta = \frac{1}{6} \left(2(a_1 + a_2) - \sum_{i=1}^2 \sqrt{(3p_B - 2a_i - w_1 - w_2)^2 + 4w_j(a_i - w_i)} \right)$$

Now, using Eq. 9 for a lower bound of p_B , the bundle discount can be written (with $\epsilon \geq 0$ as)

$$\delta = \frac{1}{6} \left(2(a_1 + a_2) - \sum_{i=1}^2 \sqrt{\left((a_j - a_i) - \frac{a_1(a_2 - w_2) + a_2(a_1 - w_1)}{2(a_1 + a_2)} + \epsilon \right)^2 + 4w_j(a_i - w_i)} \right).$$

Dropping the less significant term $4w_j(a_i - w_i)$ and ϵ , this reduces to

$$\delta \leq \frac{1}{6} \left(2(a_1 + a_2) + \frac{a_1(a_2 - w_2) + a_2(a_1 - w_1)}{(a_1 + a_2)} \right).$$

Hence the discount is highest when $w_1=0=w_2$, and most sensitive to w_1 because $a_2 \geq a_1$. ■

Proof of Lemma 1. Differentiating Eq. 16 with respect to p_1 and p_2 we get the two first-order conditions corresponding to $i=1, 2$ below, isolating p_i from p_B .

$$6p_i = (3p_B + 2a_i - \tilde{\phi}) + \sqrt{(3p_B - 2a_i)^2 + (8a_i\tilde{\phi} - 6p_B\tilde{\phi} + \tilde{\phi}^2)}. \quad (18)$$

While the equation for each $i=1, 2$ has two solutions—corresponding to the positive and negative signs of the square root term, we eliminate the positive sign from consideration by a process of contradiction. Consider the positive sign, and split the analysis into two cases.

1. $p_B < a_i$: Rewrite the term inside the square root term of Eq. 18 as $(3p_B - 2a_i + \tilde{\phi})^2 + 12(a_i - p_B)$. Since $12(a_i - p_B)$ is positive, dropping it from the equation has the implication that $6p_i > (3p_B + 2a_i - \tilde{\phi}) + (3p_B - 2a_i + \tilde{\phi}) = 6p_B$ which is impossible because $p_i \leq p_B$.
2. $p_B \geq a_i$: Again, consider the term inside the square root rewritten as $(3p_B - 2a_i + \tilde{\phi})^2 +$

$12(a_i - p_B)$. Now, rewriting $(3p_B - 2a_i + \tilde{\phi})$ as $(3(p_B - a_i) + (a_i + \tilde{\phi}))$ and then resolving the entire square root term, it is easy to show that it exceeds $\sqrt{(a_i + \tilde{\phi})^2}$. Taking the positive square root implies that the RHS exceeds $(3p_B + 2a_i - \tilde{\phi} + (a_i + \tilde{\phi}))$ which exceeds $6a_i$ because $p_B \geq a_i$, again violating the constraint $p_i < a_i$.

This eliminates the positive sign, leaving a unique statement of p_i as a function of p_B . ■

Proof of Proposition 3. First, consider what a partial bundling solution would look like if it were optimal. Setting $p_2 = p_B$ into Eq. 17a (Lemma 1) we get $p_B = a_2$. (Note that this means that the real price $\tilde{p}_B = a_2 - \tilde{\phi}$). Next, to identify the conditions under which such a solution would be optimal, rewrite the profit function in terms of p_B alone (applying Lemma 1) and compute the second derivative of the profit function against p_B . This produces $\frac{\partial^2 R}{\partial p_B^2} = -2a_1 - 2a_2 + 3p_B - Z$. Now, for the boundary solution (partial bundling) to be the global optimal, the second derivative of the objective function must be rising at the boundary $p_B = a_2$. Plugging this into the second derivative, the condition is that partial bundling is optimal when $a_2 > 2a_1 + \tilde{\phi}$.

Next, to compute the optimal value of p_1 under the partial bundle solution, plug $p_B = a_2$ into the p_1 value from Lemma 1; a rearrangement of terms yields the RHS given in Proposition 3. ■

Bhargava 2013 developed an exact analytical solution for a variant of the mixed bundling optimization problem posed in Eq. 16, one with $\tilde{\phi} = 0$ (i.e., goods with additive valuations). That solution is not directly applicable here because of our interest in $\tilde{\phi} > 0$. However,