

Price Competition in Multi-sided Markets*

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Abstract

This paper studies a general model of price competition among platforms offering differentiated services in multi-sided markets. We incorporate a general form of both within-side and cross-side externalities into a discrete choice model of random utility maximization by consumers on each side of the markets. We consider a two-stage game in which the platforms choose prices (or user fees) simultaneously in the first stage, followed by consumers on all sides simultaneously deciding which platform to join (single-homing) in the second stage. We show that in a symmetric setting with full market coverage, there exists a symmetric equilibrium in prices and the equilibrium price on each side follows a simple rule: The price equals the cost, plus a mark-up due to product differentiation, minus a subsidy due to cross-side externalities. The subsidy to each side accounts for the degree of the aggregate marginal externalities of that side imposed on all the other sides. As competition among platforms increases, both the product differentiation effect and the cross-subsidy are shown to decrease. As such, the price on one side can decrease while the price on another side may increase with the number of platforms. We also discuss the incentives for platforms to merge and the extent of excessive free entry of platforms into the markets as compared to the social optimum. We further compare uniform pricing rule with discriminatory pricing across different sides of the markets and find that the average price across sides under the discriminatory pricing is higher than the uniform price when the externalities are small or when the number of platforms is large. The impacts of consumers' outside options on the equilibrium prices are also studied.

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1 Introduction

In this paper, we study a general model of price competition among platforms providing differentiated services to customers in multi-sided markets. We incorporate a general form of cross-side externalities into a discrete choice model of random utility maximization by customers on each side of the markets and study the impacts of platform competition on the equilibrium prices and customer participation rates.

There are numerous examples of platforms providing services to allow multiple groups (or sides) of customers to interact, where one group's benefit from joining a platform depends on the sizes of the other groups that also join the platform (i.e., cross-side externalities). Examples of this include Internet companies such as Alibaba, eBay, Facebook, Uber. Alibaba and eBay facilitate e-commerce transactions between buyers and sellers; Uber connects riders and drivers through its ride-sharing platform; Facebook can be viewed as a three-sided platform with users, advertisers, and publishers (or content providers). The presence of cross-group externalities provides a rationale for the platforms to charge differential prices across different sides, often in the form of subsidizing some groups of customers. The huge success of these platform business models inevitably invite competition from entrants. Didi Chuxing and Uber China fiercely competed for several years to induce passengers and taxi drivers to adopt their own platforms; Uber and Lyft have been aggressively competing with local cab companies in the U.S. markets. The popularity of online advertisement exchange platforms (Ad Exchanges) has attracted major competing operators including Google's DoubleClick, Microsoft's AdECN, Yahoo's RightMedia and Facebook's FBX, through which users, advertisers, and publishers (or content providers) interact with each other.

How do platforms compete for multiple groups or sides of potential customers? Would an increase in platform competition reduce cross-subsidization and increase the prices (or user fees) and welfare? How would the presence of distinct multiple sides of customers affect the equilibrium outcome in terms of both discriminatory and uniform pricing across sides? There is now a large literature on platform provisions in multi-

sided markets.¹ With some exceptions, most of the papers in this literature focus on two sides, duopolistic platforms, and a Hotelling specification of product differentiation. To answer the above questions and particularly study the impacts of platform competition on the prices and welfare in many applications, we need to allow an arbitrary number of platforms, a more general form of product differentiation, and possibly more than two sides. The purpose of this paper is to provide a general model incorporating both own-side and cross-side externalities into a discrete choice model of random utility maximization by customers on each side of the markets. We follow closely the framework of [Armstrong \(2006\)](#) and consider a two-stage game in which $n \geq 2$ platforms choose prices (or user fees) simultaneously in the first stage, followed by $s \geq 1$ groups of customers simultaneously deciding which platform to join (single-homing or one-stop shopping) in the second stage.

The decision by a customer on each side of which platform to join is formulated as a discrete choice maximization of random utilities. In the presence of own-side and cross-side externalities, the demand system in our setting is interdependent across platforms and is implicitly determined by a system of nonlinear equations. While the existence of participation equilibrium (PE) follows from Brouwer's fixed point theorem, multiple equilibria may exist due to the externalities across sides. We provide sufficient conditions to ensure the uniqueness of PE for any price profile, and such conditions hold as long as the extent of externalities is not so large, relative to the degree of product differentiation among platforms.

To show the existence of the price equilibrium in the first stage involves two technical challenges. First, as mentioned above, the demand system does not have an explicit expression, which implies that the changes in a platform's demand respect to his own prices (i.e., the Jacobian matrix) and the platform's first-order conditions for profit maximization cannot be explicitly determined. Second, the conditions to deter global deviations in prices by each platform (i.e., the second-order conditions) are not easy to specify.² We provide an approach of transforming the profit maximization problem in

¹ The seminar work on two-sided markets starts with [Rochet and Tirole \(2003, 2006\)](#), [Caillaud and Jullien \(2003\)](#) and [Armstrong \(2006\)](#), which provides a basic framework for studying pricing schemes under both monopoly and duopoly structures. See [Rysman \(2009\)](#) for a survey of the early literature on the economics of two-sided markets. [Weyl \(2010\)](#) provides a theory of multi-sided markets under monopoly using nonlinear insulating tariffs. [Jullien \(2011\)](#) studies Stackelberg price competition in a multi-sided market. For examples of more recent studies and references, see [White and Weyl \(2016\)](#) and [Jullien and Pavan \(2016\)](#).

²Without externalities, each platform's profit maximization problem in prices is separable across sides

prices into one in quantities, and this allows us to circumvent the two technical challenges mentioned above.

In the symmetric setting of platforms with full market coverage, in Theorem 1 of our paper, we prove the existence of a symmetric equilibrium in which the price on each side i follows a simple, decomposable formula: The price equals the cost (c), plus a mark-up ($M_i(n)$) due to market power associated with product differentiation, minus a subsidy ($\eta_i(n)$) due to cross-side externalities, i.e.,

$$p_i^*(n) = c_i + M_i(n) - \eta_i(n),$$

where both the production differentiation effect and subsidy effect depend on the number of platforms n . To support this symmetric equilibrium, we check the incentive of a platform, say platform 1, to deviate from the proposed prices. Instead of writing the deviating profit as a function of prices, which does not have an explicit expression, we rewrite it as a function of quantities by considering his inverse demand system conditioning on other platforms' prices. Intuitively, given the other platforms' prices at the symmetric equilibrium, for platform 1 to implement demands $\mathbf{q}^1 = (q_1^1, \dots, q_s^1)$, the additional price he could charge on each side i is equal to the inverse demand without externalities, $H_i^{-1}(1 - q_i^1)$, adjusted by the differences in the externalities that a customer on each side i enjoys between platform 1 and any other platform who shares the remaining market with the $n - 1$ platforms equally. The deviating profit as a function of quantities, is separable in terms of the impacts of the distributions and the externalities, and separate assumptions on these terms can be easily made to guarantee its concavity.

The characterization of the equilibrium prices allows us to study the impacts of platform competition on prices, profits, and welfare. We raise the following three questions. The first is how prices change with respect to competition. Based on the explicit pricing formulae, we show that the market power mark-up due to product differentiation, M_i , is monotonically decreasing in n under the log-concavity assumption on the distributions, which is satisfied for many commonly used distributions. The subsidy to each side, η_i , accounts for different degrees of total externalities of a group on all the other groups, which is shown to be decreasing in the number of platforms under fairly weak conditions. The net effect of competition on prices depends on the degree of product

of the markets, and for each side the demand function, explicitly derived from the discrete choice model, can be shown to be log-concave in its own price under certain assumptions on the distributions of the random utilities (e.g., log-concavity of the distributions, see [Caplin and Nalebuff \(1991\)](#)), and hence the profit on each side is log-concave in price and the second-order condition is satisfied.

differentiation and the size of externalities. To illustrate the net effect we consider a class of distributions (i.e., Gumbel distributions) and linear forms of externalities and find that as competition increases, the price on one side can decrease, but the price on another side may increase. In other words, due to cross subsidization, the prices charged to certain sides can actually increase as competition among platforms goes up.

Second, how would equilibrium profits and total surplus change with respect to platform competition, and in particular, would free entry lead to excessive entry as compared to the social optimum when there is a fixed cost of entry? In the presence of cross-side externalities, the equilibrium profit per platform may not necessarily decrease with n . Except for the price effects, the customers prefer varieties, so the expected consumer surplus may increase or decrease with n . The total surplus may not necessarily increase with the number of platforms since the magnitudes of the externalities decrease even though the price effects do not affect the total surplus. We provide sufficient conditions for excessive entry, which rely only on the distributions, but not on the externality functions. Under Gumbel distributions, we show that there is always excessive entry into the markets, regardless of the degree of externalities, although both the socially optimal number of platforms and the equilibrium number of platforms under free entry are affected by cross-subsidization. A third question we ask is whether platforms have incentives to raise prices through a merger. Utilizing the equilibrium conditions for platform profit maximization, we are able to demonstrate that the merging platforms have incentives to increase their prices marginally on every side of the markets. Furthermore, we find the magnitudes of cost reductions on the merging platforms that make the equilibrium prices unchanged before and after the merger are proportional to the pre-merger price-cost margins.

Moreover, we compare uniform pricing with discriminatory pricing across sides. The price-cost mark-up under the symmetric uniform price is determined by the familiar demand semi-elasticity, evaluated aggregately over all sides. The product differentiation effect and subsidy effect under the uniform pricing cannot be isolated as in the case of discriminatory pricing. Since the total welfare remains unchanged under the two pricing rules, platforms and customers have opposite preferences, depending on the relative magnitudes of the average discriminatory prices across sides and the uniform price. We show that the average discriminatory prices across sides is greater than the uniform price and the platforms prefer discriminatory pricing over uniform pricing rule when the number of platforms is large or when the degree of externalities is small.

Finally, we explicitly study the impacts of outside options, mainly focusing on two-

sided markets with Gumbel distributions. In the presence of explicit outside options, a full characterization of the equilibrium is available, but the equilibrium prices do not have the decomposable three-term formulae as in the case of no outside options. Nevertheless, we show that the impacts of outside options is small when there is sufficient competition among platforms.

The remainder of this paper is organized as follows. Section 2 introduces the model and assumptions. Section 3 characterizes the equilibrium outcome of the two-stage game. The effects of platform competition on prices and welfare are presented in Section 4, and some insights into merger analysis are discussed in Section 5. Section 6 compares the equilibrium outcomes under discriminatory pricing and uniform pricing rules. Section 7 studies the impacts of outside options, and Section 8 concludes. All the proofs are presented in Appendix.

2 Model

There are n platforms competing for customers from s sides by charging prices (or user fees), $n \geq 2$ and $s \geq 1$. Let $\mathbf{p}^k = (p_1^k, \dots, p_s^k)$ denote the prices charged by platform k . On each side, there is a continuum of customers with measure 1. The utility function of a customer on side $i \in \mathcal{S} := \{1, 2, \dots, s\}$ from joining platform $k \in \mathcal{N} := \{1, 2, \dots, n\}$ is

$$u_i^k = v_i^k + \phi_i(\mathbf{x}^k) - p_i^k + \epsilon_i^k. \quad (1)$$

Here v_i^k denotes the intrinsic valuation. Since we focus on ex ante symmetric platforms, we set $v_i^k = v_i$ for any k . Let $\mathbf{x}^k = (x_1^k, \dots, x_s^k)$ denote the demand profile of platform k . The function ϕ_i is a mapping from $[0, 1]^s$ to \mathbf{R} , and $\phi_i(\mathbf{x}^k)$ captures the externalities, enjoyed by customer on side i from all other sides (possibly side i itself) in platform k . The term ϵ_i^k is random utility representing customer preference characteristics, product characteristics, functional form misspecification, and so on.³ In our specification (1), we focus on the price as user fee that is not conditional on the participation of customers on any side.⁴

³See [Perloff and Salop \(1985\)](#), [Caplin and Nalebuff \(1991\)](#) and [Anderson et al. \(1992\)](#) for examples of interpretations of the random utility term.

⁴[Armstrong \(2006\)](#) considers a platform's two-part tariff on each side conditional on the participation of the other side on the same platform. [White and Weyl \(2016\)](#) allow general nonlinear tariffs that are conditional on the participations of customers on all platforms.

Let $\mathbf{P} = (\mathbf{p}^1, \dots, \mathbf{p}^n)$ denote the price profile and \mathbf{p}^{-k} denote all the prices except of platform k . Given \mathbf{P} and demand profile $\mathbf{X} = (\mathbf{x}^1, \dots, \mathbf{x}^n)$, platform k 's profit is

$$\Pi^k(\mathbf{p}^k, \mathbf{p}^{-k}; \mathbf{X}) = \sum_{i \in \mathcal{S}} (p_i^k - c_i) x_i^k, \quad (2)$$

where c_i is the marginal cost of serving a customer on side i , which is assumed to be identical across platforms.

We further make the following assumptions:

First, our model of customer behavior is based on a discrete choice model with random utility. Each customer on side i has a unit demand, she joins at most one platform (single homing). Moreover, we focus on full market coverage by assuming away outside options. As such, each customer on side i participates in one and only one platform. An alternative modeling assumption is that there is indeed an outside option which yields utility level $u_i^0 = v_i^0 + \epsilon_i^0$ on each side i , but the intrinsic value v_i is sufficiently large as compared with v_i^0 so that customers always opt out the outside option. Later in Section 7, we explicitly model the impact of non-trivial outside options on equilibrium prices.

Second, for our main result in the next section, we make two assumptions on random utilities. We assume that across different sides $i \neq j \in \mathcal{S}$, random variables $\{\epsilon_i^1, \epsilon_i^2, \dots, \epsilon_i^n\}$ are independent from $\{\epsilon_j^1, \epsilon_j^2, \dots, \epsilon_j^n\}$.⁵ Within each side i , we assume that $\epsilon_i^1, \epsilon_i^2, \dots, \epsilon_i^n$ are *symmetrically distributed*, i.e., their joint distribution is invariant under any permutation of the order of these n random variables.⁶

For certain comparative statics results in Section 4, we consider the case of independent and identically distributed (IID) shocks and conditionally independent and identically distributed (CIID) shocks, respectively. For the case of IID, we assume that $\epsilon_i^k \sim \text{IID } F_i(\cdot), \forall k \in \mathcal{N}$ (side-specific). Moreover, we assume that F_i is continuous and differentiable on the support $[a_i, \bar{a}_i]$ with continuous probability density function (PDF) f_i . Beyond these, we do not impose any functional form restrictions on F_i , although the following examples are frequently adopted for illustrations in our paper: Type I Extreme value distribution (double exponential or Gumbel distribution); Normal distribution;

⁵See [Jullien and Pavan \(2016\)](#) for a model of platform competition in two-sided markets with correlated customer preferences between the two sides. Assuming full market coverage and symmetry between two platforms, they study the impacts of correlation between customers' information on the equilibrium prices and participation rates.

⁶Our assumption of within-side symmetry could accommodate some spatial models of price differentiation such as the Hotelling specification with uniform distribution of consumer location (for $n = 2$) and the "Spokes" model (for any $n \geq 2$, see [Chen and Riordan \(2007\)](#)).

Uniform; Exponential distribution. For the case of CIID, we assume that conditional on τ_i , $\epsilon_i^k \sim \text{IID } F_i(\cdot|\tau_i), \forall k \in \mathcal{N}$, where τ_i is distributed according to $G_i(\cdot)$. When G_i is degenerate, the model reduces to the IID case.

Third, we assume that $\phi_i(\mathbf{x})$ is continuously differentiable in \mathbf{x} with $\phi_i(\mathbf{0}) = 0$ (normalization). The most common specification of ϕ_i is linear, i.e., $\phi_i(x_1, x_2, \dots, x_s) = \sum_{j \in \mathcal{S}} \gamma_{ij} x_j$. The parameters γ_{ij} is not necessarily nonnegative, i.e., negative externalities (congestion effect) are also allowed. Our specification could incorporate within-side network externalities: $\frac{\partial \phi_i}{\partial x_i} \neq 0$, which is natural in some applications. Moreover, we are not limited to linear functional forms, and increasing (decreasing) return to scale of externalities are also permitted, for example, $\phi_i(x_1, x_2, \dots, x_s) = \sum_{j \in \mathcal{S}} \gamma_{ij} \rho(x_j)$ with $\rho' > 0, \rho'' \geq 0 (\rho'' < 0)$; or $\phi_i(x_1, x_2, \dots, x_s) = \mu(\sum_{j \in \mathcal{S}} \gamma_{ij} x_j)$, with $\mu' \geq 0, \mu'' > 0 (\mu'' < 0)$.

We study the subgame perfect equilibrium of the following two-stage game: Platform providers simultaneously choose their prices first, followed by customers simultaneously deciding which platform to participate.

We use the following notation: Let $\Psi_{ij}(\mathbf{x}) = \frac{\partial \phi_i(\mathbf{x})}{\partial x_j}$, and $\Psi(\mathbf{x}) = (\Psi_{ij}(\mathbf{x}))_{1 \leq i, j \leq s}$ denote the *marginal* externalities matrix at \mathbf{x} . When ϕ_i takes the linear form, i.e., $\phi_i(\mathbf{x}) = \sum_{j \in \mathcal{S}} \gamma_{ij} x_j$, we let $\Gamma = (\gamma_{ij})_{s \times s}$ be the linear externality matrix, i.e., $\Psi(\mathbf{x}) = \Gamma, \forall \mathbf{x}$.

For given vector $\mathbf{w} = (w_1, \dots, w_n)$, we let $\text{Diag}(\mathbf{w}) = \text{Diag}(w_1, \dots, w_n)$ denote the n by n diagonal matrix with w_i on its ii -entry. For a matrix \mathbf{M} , its transpose is denoted as \mathbf{M}' . The n by n identity matrix is denoted as \mathbf{I}_n , while $\mathbf{0}$ denotes the zero matrix with suitable dimension. The column vectors of 1s of length s is denoted as $\mathbf{1}_s = (1, 1, \dots, 1)'$.

3 Equilibrium analysis

In this section we determine the subgame perfect equilibrium of the two-stage game. By backward induction, we first study the equilibrium in the second stage, characterizing customers' simultaneously participation decisions. For each fixed price profile \mathbf{P} , we call demand profile $\mathbf{X}(\mathbf{P})$ a participation equilibrium (PE) associated with \mathbf{P} if it is a pure strategy Nash Equilibrium in the second stage.

Given \mathbf{P} , each customer on side i joins the platform that yields the highest utility, as specified in (1). Clearly, $\mathbf{X}(\mathbf{P})$ is a PE associated with \mathbf{P} if and only if the following equations simultaneously hold:

$$\begin{aligned} x_i^k(\mathbf{P}) &= \Pr(v_i + \phi_i(\mathbf{x}^k(\mathbf{P})) - p_i^k + \epsilon_i^k \\ &\geq \max_{t \neq k} \{v_i + \phi_i(\mathbf{x}^t(\mathbf{P})) - p_i^t + \epsilon_i^t\}) \end{aligned} \quad (3)$$

for any $i \in \mathcal{S}, k \in \mathcal{N}$.

In other words, the demand system $\mathbf{X}(\mathbf{P})$ is interdependent across platforms and implicitly determined by the above system of equations. Applying *Brouwer's fixed point theorem*, we see that the system (3) has at least one solution.

Proposition 1. *For any price profile \mathbf{P} , there exists at least one participation equilibrium.*

Due to cross-group externalities, the uniqueness of PE is not guaranteed.⁷ Nevertheless, sufficient conditions can be derived to ensure uniqueness. Intuitively, when the degree of externalities is small, relatively to the dispersion of customer preferences for differentiated products (or the noise distributions), there exists a unique PE. To illustrate, we focus on the linear form of externalities.

For $n = 2$ and general distributions, we apply the contraction-mapping theorem and obtain the following sufficient condition for uniqueness:⁸

$$2\{\max_{\theta_i} h_i(\theta_i; 2)\} \left\{ \sum_{j \in \mathcal{S}} |\gamma_{ij}| \right\} < 1, \forall i \in \mathcal{S},$$

where h_i is defined in the next paragraph.⁹ When $n \geq 2$ and ϵ_i^t is IID according to Gumbel distribution with parameter β_i , the following condition is sufficient:

$$\max_{i \in \mathcal{S}} \left\{ \sum_{j \in \mathcal{S}} |\gamma_{ij}| \right\} < 2 \min(\beta_1, \dots, \beta_s).$$

Note that this condition does not rely on n .

We now characterize the price equilibrium in the first stage. Let $H_i(\theta; n)$ and $h_i(\theta; n)$ denote the CDF and PDF of the random variable $\epsilon_i^1 - \max(\epsilon_i^2, \dots, \epsilon_i^n)$, respectively.¹⁰

⁷See [Brock and Durlauf \(2001, 2002\)](#) for discussions on possible multiple equilibria in a single-sided market with externalities. Multiple equilibria also arise in coordination games with strong complementarity, see [Carlsson and Van Damme \(1993\)](#); [Morris and Shin \(1998, 2003\)](#); [Hellwig \(2002\)](#); [Vives \(2005\)](#).

⁸When $\gamma_{ij} \geq 0$, alternative condition is given by:

$$2\lambda_{\max}(\mathbf{\Gamma}) \times \max_{i \in \mathcal{S}} \left\{ \max_{\theta_i} h_i(\theta_i; 2) \right\} < 1,$$

where $\lambda_{\max}(\mathbf{\Gamma})$ is the largest eigenvalue of nonnegative matrix $\mathbf{\Gamma}$, which also equals the spectral radius of $\mathbf{\Gamma}$ by Perron-Frobenius Theorem. For $n \geq 2$, similar conditions can be derived but are more involved.

⁹See [Appendix B](#) for details.

¹⁰The exact formula of H_i and h_i depends on the specification of random shocks. For Gumbel distribution, H_i is a logistics function, see [\(24\)](#). For Hotelling specification with uniform distribution of consumer location ($n = 2$), H_i is linear (see [Example 1](#) on page 11). For presenting our main result we impose mild restrictions on H_i . For comparative static analysis in the next section, we will focus on the case of CIID and several examples of the IID case.

For a given vector $\mathbf{a} \in \mathbf{R}^s$, define the following multi-variable function $R : [0, 1]^s \rightarrow \mathbf{R}$

$$R(\mathbf{z}; \mathbf{a}) := \sum_{i \in \mathcal{S}} z_i H_i^{-1}(1 - z_i; n) + \sum_{i \in \mathcal{S}} z_i \left\{ \phi_i(\mathbf{z}) - \phi_i\left(\frac{\mathbf{1}_s - \mathbf{z}}{n - 1}\right) \right\} + \mathbf{z} \cdot \mathbf{a}. \quad (4)$$

We assume the following regularity condition on R .

Assumption 1. For any $\mathbf{a} \in \mathbf{R}^s$, $R(\mathbf{z}; \mathbf{a})$ is concave in $\mathbf{z} \in [0, 1]^s$.

Notice that the concavity of $R(\mathbf{z}; \mathbf{a})$ in \mathbf{z} is clearly not affected by \mathbf{a} , as the third term $\mathbf{z} \cdot \mathbf{a}$ in (4) is linear in \mathbf{z} . An equivalent statement of Assumption 1 is that for some \mathbf{a} , $R(\mathbf{z}; \mathbf{a})$ is concave in $\mathbf{z} \in [0, 1]^s$. Later we show that the R function, for appropriately chosen \mathbf{a} , becomes the profit function of a platform, if the platform uses quantities $\{z_1, \dots, z_s\}$ as his strategy variables, instead of prices. Essentially, Assumption 1 requires that platform's revenue function, hence profit function, be concave in quantities. This assumption can be easily verified once the distributions of random utilities and the functions of externalities are given.

To state our main result on the equilibrium prices, we define

$$M_i(n) := \frac{1 - H_i(0; n)}{h_i(0; n)}, \quad \text{and} \quad \eta_i(n) := \frac{1}{n - 1} \sum_{j \in \mathcal{S}} \Psi_{ji}(\mathbf{x})|_{\mathbf{x} = \frac{1}{n} \mathbf{1}_s}, \quad i \in \mathcal{S}$$

where M_i represents a semi-elasticity of price and measures the degree of product differentiation among platforms on side i , which is completely determined by the distributions of random utilities, and η_i measures the impact of externalities that side i generates to all other sides when the markets are equally divided among platforms.

Theorem 1. Under full market coverage and Assumption 1, there exists a subgame perfect Nash equilibrium with the outcome that all the platforms charge the same prices $\mathbf{p}^* = (p_1^*, \dots, p_s^*)$ in the first stage and the equilibrium market demand is $\mathbf{x}^* = \frac{1}{n} \mathbf{1}_s$ for each platform in the second stage, where¹¹

$$p_i^*(n) = c_i + M_i(n) - \eta_i(n), \quad i \in \mathcal{S} \quad (5)$$

Theorem 1 shows that in our general setting, the equilibrium prices follow a simple formula. The price charged to each side consists of three additively separable terms: the cost, a mark-up due to product differentiation, and a subsidy due to externalities. Each term in the pricing formula represents a separate economic factor and is explicitly

¹¹Under the sufficient condition for the uniqueness of PE, the symmetric pricing equilibrium is unique, and takes the form given by (5).

determined by the primitives of the model.¹² The term $M_i(n)$ is the standard measure of market power of the oligopolistic firms offering differentiated products and is determined by the distributions of random utilities (see [Perloff and Salop \(1985\)](#)), which depends explicitly on the number of platforms in the case of symmetry.

What is interesting here is the determination of the subsidy term. Note that the participation by side i customer generates externalities to all sides (including her own side). When a platform, say platform 1, attracts additional Δ_i users on side i , the externalities enjoyed by any other side j from joining platform 1 is enhanced by $\Psi_{ji}\Delta_i$, while the externalities from joining any other competing platform is reduced by $\Psi_{ji}\frac{\Delta_i}{n-1}$, since each of the remaining $n-1$ platforms would lose $\frac{\Delta_i}{n-1}$ on side i due to a business stealing effect. The difference in willingness to pay by each side j customer then equals $\Psi_{ji}(1 + \frac{1}{n-1})\Delta_i$. The additional profit that the platform could capture from attracting additional user Δ_i is thus equal to the sum of the marginal externalities multiplied by the equilibrium market share x_j^{*1} that side i imposed on all sides. In other words, to encourage customer from side i to participate, in equilibrium platform 1 needs to lower his price by an amount equal to the aggregate marginal profits:

$$\sum_{j \in \mathcal{S}} \Psi_{ji} \times (1 + \frac{1}{n-1}) x_j^{*1},$$

which is the subsidy in Theorem 1, where $x_j^{*1} = 1/n$ is the equilibrium market share of platform 1.

Example 1. To illustrate Theorem 1, we consider a special case of our model. In his model of platform competition ($n = 2$) in two-sided markets ($s = 2$), [Armstrong \(2006\)](#) uses a Hotelling specification with uniform distribution of consumer location. The equilibrium prices in his setting are given by

$$p_1^* = c_1 + t_1 - \alpha_2, \quad p_2^* = c_2 + t_2 - \alpha_1,$$

where t_1 and t_2 are the product differentiation (or transport cost) parameters, and α_1 and α_2 are the degrees of cross-group externalities enjoyed by side 1 and 2, respectively (see equation (12) in Proposition 2 in his paper). This pricing formula is the same as

¹²The simplicity of the pricing formula is a joint product of symmetry, one-stop shopping, full market coverage, and the additive separability of the consumer payoff, and does not rely on the functional forms of externalities and the distributions of random utilities. We will show in Section 7 that in the presence of explicit outside options, the markets are not fully covered, and the equilibrium prices cannot easily decomposed into the three separate effects.

ours in (5). Indeed, in this Hotelling specification of product differentiation, the random utilities can be represented by $\epsilon_i^1 = -t_i \tilde{\epsilon}_i$, $\epsilon_i^2 = -t_i(1 - \tilde{\epsilon}_i)$ with $\tilde{\epsilon}_i$ being uniformly distributed on the unit interval $[0, 1]$. It follows that $H_i(\theta; 2) = 1/2 + \frac{\theta}{2t_i}$, $\theta \in [-t_i, t_i]$, and hence $(1 - H_i(0; 2))/h_i(0; 2) = t_i$, $i = 1, 2$. Moreover, in this setting, R is a quadratic function of \mathbf{z} with Hessian matrix $2 \begin{bmatrix} -2t_1 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 & -2t_2 \end{bmatrix}$, and Assumption 1 means that this Hessian matrix is negative definite, or equivalently $4t_1 t_2 > (\alpha_1 + \alpha_2)^2$, which is what [Armstrong \(2006\)](#) assumes (see equation (8) in his paper).

What is implicitly assumed in the Hotelling duopoly model in [Armstrong \(2006\)](#) is that the intrinsic value v_i is relatively large so that customers always join at least one of the two platforms, and hence the market are always fully covered. This argument is applicable to our general setting with an outside option imposed on each side, and hence the symmetric pricing in Theorem 1 remains to be an equilibrium if (i) intrinsic values are sufficiently large, relatively to the values of the outside options, and (ii) the supports of the distributions of random utilities are bounded.

Before further analyzing the above pricing formula, we provide a sketch of the proof of Theorem 1. We need to show that there is no incentive for any platform to deviate unilaterally from the proposed equilibrium price vector \mathbf{p}^* in (5). To check the incentive of a platform, say platform 1, to deviate from \mathbf{p}^* to \mathbf{p}^1 , we need to pin down his deviating profit as a function of \mathbf{p}^1 . One challenge is that due to the cross-side externalities, the demand system for platform 1's services is implicitly defined by (3), and hence the deviating profit function does not have an explicit expression. Instead, we rewrite platform 1's deviating profit as a function of quantities \mathbf{q}^1 by considering his inverse demand system. Intuitively, given other platforms' choices of \mathbf{p}^* , platform 1 behaves as a monopolist, and choosing prices \mathbf{p}^1 by the monopolist is equivalent to choosing quantities \mathbf{q}^1 , subject to the demand system conditional on other platforms' prices.

Assume that the $n - 1$ platforms have the same demand when charging the same price \mathbf{p}^* . This together with full market coverage assumption implies that whatever quantities \mathbf{q}^1 platform 1 offers, the remaining market is shared equally among the $n - 1$ platforms, with each taking $\frac{1_s - \mathbf{q}^1}{n-1}$. It follows that the (conditional) inverse demand system that platform 1 faces has an explicit expression: The extra willingness to pay by each side i , $p_i^1 - p_i^*$, equals $H_i^{-1}(1 - q_i^1)$ (the inverse demand without externality), adjusted by $\phi_i(\mathbf{q}^1) - \phi_i(\frac{1_s - \mathbf{q}^1}{n-1})$ (the difference in the externalities that customer i enjoys between platform 1 and any other platform). We can then reformulate the deviating profit as $R(\mathbf{q}^1; \mathbf{p}^* - \mathbf{c})$ in (4). Deviating is not profitable if $R(\mathbf{q}^1; \mathbf{p}^* - \mathbf{c})$ is maximized

at the equilibrium demand profile, $\mathbf{q}^1 = \frac{1}{n}\mathbf{1}_s$. The first-order conditions for this profit maximization problem hold when \mathbf{p}^* satisfies (5). Moreover, given that R is concave in \mathbf{q}^1 by Assumption 1, the first-order conditions are necessary and sufficient for profit maximization. Therefore, there is no profitable deviation by platform 1.

Moreover, to support the equilibrium outcome given in Theorem 1, we need to specify participation equilibrium (PE) for each price profile \mathbf{P} . As discussed previously, there are sufficient conditions under which for each price profile \mathbf{P} , there is a unique PE. However, in general there might be multiple participation equilibria. We have made the following reasonable selection: on the equilibrium path as every platform offers \mathbf{p}^* , we pick PE such that markets are shared equally among these n platforms; off the equilibrium path when only one platform deviates to a different price \mathbf{p}^1 while remaining $n - 1$ platforms charge \mathbf{p}^* , we focus on the *semi-symmetric* PE such that the remaining $n - 1$ platforms charge the same price receive the same demand. Such *semi-symmetric* PE always exists and can be characterized by fixed-point conditions, similar to, but simpler than (3). When more than 2 platforms deviate from \mathbf{p}^* , simply pick any PE.

One advantage with transforming a platform's profit maximization in prices into the one in quantities is that sufficient conditions for maximization can be easily specified and checked.¹³ Indeed, since R function in (4) is separable in terms of the impacts of the distributions and externalities, separate assumptions on these functions can be made to guarantee Assumption 1. This is not possible when considering profit maximization in prices, since there is no explicit demand system in this general setting of multi-sided markets. Assumption 1 can be replaced by simpler but stronger assumptions. Suppose $1 - H_i(\theta_i; n)$ is log-concave in θ_i . Then $z_i H_i^{-1}(1 - z_i; n)$ is concave in z_i (as shown in Appendix C). As a consequence, the first term in $R(\mathbf{z}, \mathbf{a})$, $\sum_i z_i H_i^{-1}(1 - z_i; n)$, is concave in $\mathbf{z} \in [0, 1]^s$. In other words, without cross-side externalities, the log-concavity of each $1 - H_i$ is sufficient to imply Assumption 1. In general, the cross-side externality term ϕ_i non-trivially affects the Hessian matrix of R , and Assumption 1 roughly means that the degree of externalities is relatively small, as compared with the product differentiation effect. Under the linear form of externalities, we can instead impose the following alternative assumptions:

¹³See [Nocke and Schutz \(2016\)](#) for a model of multi-product oligopolistic competition using techniques from aggregate games. Due to the multi-sided features and cross-group externalities, the demand system in our model is not explicit and the products within each platform are complements instead of substitutes. Therefore, their approach does not apply to our model.

Assumption 2. (i) For each $i \in \mathcal{S}$, $1 - H_i(\theta_i; n)$ is log-concave in θ_i ; (ii) the matrix

$$\text{Diag}(w_1, \dots, w_s) - \frac{n}{n-1}(\mathbf{\Gamma} + \mathbf{\Gamma}')$$

is positive definite, where $1/w_i = \max_{\theta_i} h_i(\theta_i; n) > 0$.

In Appendix C, we formally show that Assumption 2 implies Assumption 1. Note that for IID random variables, Assumption (2-i) holds if for each i , the PDF $f_i(\cdot)$ is log-concave, which holds for many commonly used distributions including normal, uniform, exponential, and Gumbel distributions. Assumption (2-ii) only requires positive definiteness of a single matrix, which is simpler to check. For Gumbel distributions, (2-ii) is equivalent to

$$4\text{Diag}(\beta_1, \dots, \beta_s) - \frac{n}{n-1}(\mathbf{\Gamma} + \mathbf{\Gamma}')$$

being positive definite, which is true if the maximal eigenvalue of matrix $(\mathbf{\Gamma} + \mathbf{\Gamma}')$ is less than $4 \min\{\beta_i\}(n-1)/n$.

4 The effect of platform competition on prices and welfare

Theorem 1 reveals *decomposable pricing formula* for symmetric equilibrium prices: cost, product differentiation effect and cross subsidy, where the product differentiation effect is M_i (Perloff and Salop 1985) and the cross subsidy equals η_i , which summarizes the externalities that group i offers to all other groups. Both product differentiation effect and cross subsidy vary with the degree of competition (n), and the net effect depends on the degree of product differentiation and the size of externalities.

To study the impact of platform competition on the equilibrium prices and welfare, we first consider the case of CIID random variables. In this case, we have

$$H_i(\theta; n) = \Pr(\epsilon_i^1 - \max_{\{t=2, \dots, n\}} \epsilon_i^t \leq \theta) = \int \left(\int F_i(\theta + \xi | \tau_i) dF_i^{n-1}(\xi | \tau_i) \right) dG_i(\tau_i)$$

and $h_i(\theta; n) = \int \left(\int f_i(\theta + \xi | \tau_i) dF_i^{n-1}(\xi | \tau_i) \right) dG_i(\tau_i)$. Therefore, $H_i(0; n) = (n-1)/n$ and $h_i(0; n) = \int \left(\int f_i(\xi | \tau_i) dF_i^{n-1}(\xi | \tau_i) \right) dG_i(\tau_i)$.

Proposition 2. Suppose under the CIID, for each τ_i , $1 - F_i(\theta | \tau_i)$ is log-concave in θ . Then the product differentiation effect $M_i(n)$ is decreasing in n .

For the case of IID random variables, the effect of competition on market power is known in the literature. For instance, [Zhou \(2017\)](#) shows that under the log-concavity of $1 - F_i$, $M_i(n)$ is monotonically decreasing in n , and approaches zero as n goes to infinity, if additionally $\lim_{\theta_i \rightarrow \infty} f_i(\theta_i)/(1 - F_i(\theta_i)) = +\infty$.¹⁴ Proposition 2 extends the monotonicity of product differentiation effect to the case of CIID.

On the other hand, in the literature on multi-sided markets, not much attention has been paid to the impact of platform competition on the extent of cross-subsidies. The following Proposition provides some insights.

Proposition 3. *Suppose $\phi_i(\mathbf{x}) = \sum_{j \in \mathcal{S}} \gamma_{ij} \rho(x_j)$ with $\rho' > 0$.¹⁵ Then the equilibrium cross-subsidy equals $\eta_i(n) = \frac{\rho'(1/n)}{n-1} \bar{\gamma}_i$, where $\bar{\gamma}_i := \sum_{j \in \mathcal{S}} \gamma_{ji}$. Assume $x\rho''(x) + \rho'(x) > 0$ and $\lim_{x \rightarrow 0} x\rho'(x) = 0$.¹⁶ Then η_i decreases with n if and only if $\bar{\gamma}_i > 0$. Moreover, η_i converges to 0 as $n \rightarrow \infty$.*

Proposition 3 shows that for a general class of externalities with weakly increasing returns to scale, the magnitude of subsidy to each side decreases with n . As competition increases, the equilibrium number of users on each side decrease, and the loop effect is weaker, moreover the marginal externalities is also weaker given weakly increasing returns to scale, thus the cross-subsidy term decreases with n . Here the loop effect is equal to $1 + \frac{1}{n-1}$ as each additional user on platform 1 implies $1/(n-1)$ loss for each of the other platforms.

Moreover, note that for any two sides, i and j , $\eta_i - \eta_j = \frac{\rho'(1/n)}{n-1} (\bar{\gamma}_i - \bar{\gamma}_j)$. The relative degree of subsidies to the two sides depends on the relative marginal externalities that each side imposes on all the other sides. Suppose $c_i = c_j$ and $M_i = M_j$, then $\bar{\gamma}_i < \bar{\gamma}_j$ implies $p_i^* > p_j^*$. That is, each platform subsidizes customers on side j by charging more on side i . As competition increases, the extent of cross-subsidies decreases since $\frac{\rho'(1/n)}{n-1}$ decreases with n . In other words, an increases in competition erodes the relative degree of cross subsidies.

Propositions 2 and 3 show the monotonicity of the product differentiation effect and cross-subsidy effect with respect to the degree of competition. To illustrate the net effect

¹⁴When the support of θ_i has a upper bound θ_i^+ and $f_i(\theta_i^+) > 0$, $\lim_{\theta_i \rightarrow \theta_i^+} f_i(\theta_i)/(1 - F_i(\theta_i))$ is infinite. When θ_i is unbounded from above, the limit can be infinite (e.g, normal distributions) or finite (e.g., Gumbel and exponential distributions). See Proposition 2 and 3 and related discussions in [Perloff and Salop \(1985\)](#).

¹⁵The case with $\phi_i(\mathbf{x}) = \mu(\sum_{j \in \mathcal{S}} \gamma_{ij} x_j)$ with increasing μ is similar.

¹⁶The conditions $x\rho''(x) + \rho'(x) > 0$ and $\lim_{x \rightarrow 0} x\rho'(x) = 0$ are satisfied when ρ is convex, or when ρ is a power function ($\rho(x) = x^r, 0 < r < \infty$) or log function ($\rho(x) = \log(1 + tx), t > 0$).

of competition on prices, we consider the following example.

Example 2. Suppose ϵ_i^t is IID with exponential distribution $F_i(\theta) = 1 - e^{-\lambda_i\theta}$ and $\lambda_i > 0$. It follows that the product differentiation effect $M_i(n) = 1/\lambda_i$, which is constant in n (see [Perloff and Salop 1985](#)). With linear form of externalities, the equilibrium prices are given by

$$p_i^* = c_i + \frac{1}{\lambda_i} - \frac{1}{n-1}\bar{\gamma}_i, \quad i \in \mathcal{S}.$$

Therefore, we have an intriguing observation that the equilibrium price on each side i is increasing in the number of platforms n whenever there are positive aggregate externalities ($\bar{\gamma}_i > 0$).

For welfare analysis, we define $\delta_i(n) = \mathbb{E}[\max_{k=1,2,\dots,n} \epsilon_i^k]$ as the expected value of the maximum of n random variables. The following proposition presents the equilibrium profit for each platform, the expected consumer surplus for each side, and the total surplus (the sum of total consumer surplus and total profits).

Proposition 4. *In the symmetric equilibrium characterized by Theorem 1, the following hold. Each platform $k \in \mathcal{N}$ earns the same profit*

$$\Pi(n) = \frac{1}{n} \sum_{i \in \mathcal{S}} (M_i - \eta_i); \quad (6)$$

The expected consumer surplus for side $i \in \mathcal{S}$ equals

$$CS_i(n) = v_i - c_i - M_i + \delta_i + \eta_i + \phi_i\left(\frac{1}{n}\mathbf{1}_s\right); \quad (7)$$

The total surplus equals

$$TS(n) = \sum_{i \in \mathcal{S}} \left(v_i - c_i + \delta_i + \phi_i\left(\frac{1}{n}\mathbf{1}_s\right) \right). \quad (8)$$

Note that δ_i reflects customer's preference over variety and increases with n for the case of CIID. Without externalities, Propositions 2 and 4 indicate standard results that the equilibrium profit for each firm decreases with n while both the expected consumer surplus and total surplus increase with n . In the presence of cross-group externalities, as shown in Proposition 3, platforms have incentives to subsidize across different sides, which may cause the equilibrium prices on some sides to increase with n . Moreover, for monotonically increasing function of externalities, $\phi_i(\frac{1}{n}\mathbf{1}_s)$ decreases with n . As a result

of the impact of cross-group externalities, the equilibrium profit per platform may not necessarily decrease with n , and the expected consumer surplus and total surplus may not increase with n either.

One implication of Proposition 4 is that there might be excessive entry into the multi-sided markets even though there are cross-group externalities. To illustrate we first note that the marginal benefit of an additional platform on the total externalities enjoyed by all sides is negatively related to the total cross-subsidies:

$$\frac{\partial\{\sum_{i \in \mathcal{S}} \phi_i(\frac{1}{n} \mathbf{1}_s)\}}{\partial n} = - \sum_{i, j \in \mathcal{S}} \frac{1}{n^2} \Psi_{ij}(\frac{1}{n} \mathbf{1}_s) = - \frac{n-1}{n^2} (\sum_{i \in \mathcal{S}} \eta_i),$$

which holds for any functional form of externalities. When the aggregate externalities are positive, the total subsidies are also positive, which implies that the total benefits from externalities decreases with the number of platforms. Second, by Proposition 4, as n increases, the change in the equilibrium total surplus can be related to the per-platform profit as follows:

$$\frac{\partial TS}{\partial n} - \frac{n-1}{n} \Pi = \sum_{i \in \mathcal{S}} (\frac{\partial \delta_i}{\partial n} - \frac{n-1}{n^2} M_i),$$

where the right-hand side is independent of externalities, and depends only on the distributions of random utilities.¹⁷

Suppose there is a fixed cost $K > 0$ associated with building up a platform. There is excessive entry if the equilibrium free-entry number of platforms determined by $\Pi(n) - K = 0$ exceeds the socially optimal number of platforms that maximizes the total social surplus $TS(n) - nK$. The above discussions imply a sufficient condition for excessive entry as stated in the following Corollary.

Corollary 1. *Suppose that $TS(n) - nK$ is quasi-concave in n , and that for every side i , the distribution of random utilities satisfies the condition that $\frac{\partial \delta_i}{\partial n} - \frac{n-1}{n^2} M_i \leq 0$. Then there is excessive entry.*

The conditions in Corollary 1 are imposed only on the distributions of random utilities, not on externality functions.

To see how competition affects the prices and welfare through the trade-off between the product differentiation effect and cross-side subsidy, we consider the following class of distributions and the linear form of externalities.

¹⁷The right-hand side is always 0 for Gumbel distributions, and negative for uniform distributions.

Proposition 5. Suppose ϵ_i^t is IID according to the Gumbel distribution with parameter β_i , and linear forms of externalities. Let $\bar{\beta} = \sum_{i \in \mathcal{S}} \beta_i$ and $\bar{\gamma} = \sum_{i \in \mathcal{S}} \bar{\gamma}_i$. Then the following hold.

(i) For each $i \in \mathcal{S}$, the equilibrium price is

$$p_i^*(n) = c_i + \frac{n}{n-1} \beta_i - \frac{1}{n-1} \bar{\gamma}_i, \quad (9)$$

which decreases with n if and only if $\beta_i > \bar{\gamma}_i$.

(ii) Each platform's equilibrium profit is

$$\Pi(n) = \frac{n\bar{\beta} - \bar{\gamma}}{n(n-1)}, \quad (10)$$

which decreases with n if and only if $\bar{\gamma}/\bar{\beta} < n^2/(2n-1)$;

(iii) The total surplus is

$$TS(n) = \sum_{i \in \mathcal{S}} (v_i - c_i) + (\ln(n) + \kappa) \bar{\beta} + \frac{1}{n} \bar{\gamma}, \quad (11)$$

which increases with n . Here $\kappa \approx 0.5772$ is the Euler-Marcheroni constant.

(iv) There is always excessive entry.

Part (i) In Proposition 5 provides a simple formula for the equilibrium prices, which implies that it is feasible to have the price on one side (say i) decrease with n (when $\beta_i > \bar{\gamma}_i$) while the price on another side (say i') increase with n (when $\beta_{i'} < \bar{\gamma}_{i'}$). The latter inequalities can hold for a number of sides without contradicting with the conditions for uniqueness of participation equilibrium and sufficiency for price equilibrium.

In equilibrium, each platform balances out all sides and the profit per platform and total surplus depend only on the aggregate parameter values, $\bar{\beta}$ and $\bar{\gamma}$, as indicated in parts (ii) and (iii) of Proposition 5. As competition increases, platform profit decreases when $\bar{\gamma}/\bar{\beta} < n^2/(2n-1)$, and total surplus increases when $\bar{\gamma}/\bar{\beta} < n$. The latter condition is equivalent to positive profit for each platform. Both conditions are easily satisfied given the requirement for sufficiency of the price equilibrium. Therefore, in this setting the presence of cross-group externalities does not affect the qualitative impact of competition on the aggregate welfare.¹⁸

¹⁸The platform profit increases with the product differentiation parameter β_i , but decreases with the degree of individual externalities γ_{ij} . The total surplus increases with both β_i and γ_{ij} . It can be easily checked that the expected consumer surplus increases with γ_{ij} , but decreases with β_i when $n = 2$, and increases with β_i when $n \geq 3$.

The strength of externalities and the curvature of ϕ_i , whether ϕ_i is concave, convex or linear, clearly affect both the equilibrium number of platforms with free entry and the socially optimal one, but do not affect the comparison between these two numbers under Gumbel assumptions.¹⁹

5 Merger analysis

One application of our equilibrium analysis is to study the impact of a merger between platforms in multi-sided markets. For instance, would a merged entity have incentives to raise the prices given that all platforms engage in cross-subsidization both before and after the merger? While our analysis in the previous section relies on symmetry, it still offers some insights into this question.

In general, a key to the merger analysis is to examine the changes in demands with respect to prices both within and across platforms, which determine the own-price and cross-price elasticities as well as the diversion matrices. Given that platform k charges prices \mathbf{p}^k while other platforms choose symmetric prices \mathbf{p} , assume the demands for other platforms' services are symmetric. Then, the full market coverage implies a simple relation between the demand of platform \mathbf{x}^k and that of other platforms, i.e., $\mathbf{x}^{k'} = \frac{1_s - \mathbf{x}^k}{n-1}$, $\forall k' \neq k$. This further implies that $\frac{\partial \mathbf{x}^{k'}}{\partial \mathbf{p}^k} = -\frac{1}{n-1} \frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k}$. As a consequence, the diversion matrix from k to k' is simply

$$-\left(\frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k}\right)^{-1} \left(\frac{\partial \mathbf{x}^{k'}}{\partial \mathbf{p}^k}\right)' = \frac{1}{n-1} \mathbf{I}_s, \quad (12)$$

which is independent of \mathbf{p}^k , \mathbf{p} , and \mathbf{x}^k . The standard measure of upward pricing pressure (UPP) at the equilibrium prices is then $\frac{1}{n-1}(\mathbf{p}^* - \mathbf{c})$, which is not informative for illustrating incentives to raise or reduce prices, since some of the margins can be negative in this setting. To evaluate the incentives for the merged entity to raise prices or not, we need to compute precisely the matrix of the cross-price partial derivatives. The following Lemma presents these partial derivatives when all the prices are symmetric across platforms.

¹⁹When $\bar{\gamma} = 0$, the number of firms under free entry minus the socially optimal number equals exactly 1 (see [Anderson et al. \(1992\)](#), Table 7.1 on page 226, for this observation). In the presence of cross-side externalities with $\bar{\gamma} > 0$, the difference is strictly larger than 1.

Lemma 1. *With full market coverage, at any symmetric price \mathbf{p} with symmetric allocation $\mathbf{x} = \frac{1}{s}\mathbf{1}_n$, the following hold: For any $k, k' \in \mathcal{N}$ and $k' \neq k$,*

$$\frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k} = -\mathbf{E} \quad \text{and} \quad \frac{\partial \mathbf{x}^{k'}}{\partial \mathbf{p}^k} = \frac{1}{n-1}\mathbf{E}, \quad (13)$$

where

$$\mathbf{E}^{-1} = \text{Diag}\left\{\frac{1}{h_1(0;n)}, \dots, \frac{1}{h_s(0;n)}\right\} - \frac{n}{n-1}\mathbf{\Psi}(\mathbf{x})\Big|_{\mathbf{x}=\frac{1}{n}\mathbf{1}_s}. \quad (14)$$

Note that these partial derivatives in Lemma 1 are surprisingly simple and do not depend on the symmetric \mathbf{p} . In Appendix A, we prove a general version of Lemma 1, which presents the matrices of partial derivatives when platform k charges prices \mathbf{p}^k while other platforms choose symmetric prices \mathbf{p} . As expected, these matrices depend on $(\mathbf{p}^k, \mathbf{p}, \mathbf{x}^k)$. When $\mathbf{p}^k = \mathbf{p}$, $\mathbf{x}^k = \frac{1}{s}\mathbf{1}_n$ and the matrices greatly simplify to (13) and (14) in Lemma 1.

We now use Lemma 1 to illustrate the marginal incentive for the merged platforms to raise or reduce prices in multi-sided markets. For notational simplicity we consider platforms 1 and 2 to merge and maximize the joint profits by coordinating on their prices. The following Proposition shows that the merged entity has incentives to raise locally the prices across all sides.

Proposition 6. *At the symmetric equilibrium price \mathbf{p}^* , both merging platform 1 and 2 have strict incentive to increase price on every side of the markets in the sense that for $l = 1, 2$,*

$$\frac{\partial \{\Pi^1 + \Pi^2\}}{\partial \mathbf{p}^l} \Big|_{\mathbf{p}^k = \mathbf{p}^*, \forall k \in \mathcal{N}} = \frac{1}{n(n-1)}\mathbf{1}_s \succ \mathbf{0}.$$

Without cross-group externalities, the above result is well known for price competition with substitute goods, as the merging firms would internalize some of the benefit from increasing prices. However, in our setting it is possible that the price-cost margin $p_i^* - c_i$ is negative on some side i before the merger, since platforms engage in cross subsidization. It is not clear at the first glance that the marginal benefit to platform 2 is positive or not. However, since both platforms have the same pre-merger margins which satisfies platform 1's first-order conditions

$$\mathbf{p}^* - \mathbf{c} = -\left(\frac{\partial \mathbf{x}^1}{\partial \mathbf{p}^1}\right)^{-1} \frac{1}{n}\mathbf{I}_s.$$

Eliminating the margins, we have a simple expression for the marginal benefit of raising prices at the pre-merger symmetric equilibrium prices as follows:

$$\left\{ \frac{\partial \mathbf{x}^2}{\partial \mathbf{p}^1} \right\}' (\mathbf{p}^* - \mathbf{c}) = - \left\{ \frac{\partial \mathbf{x}^2}{\partial \mathbf{p}^1} \right\}' \left(\frac{\partial \mathbf{x}^1}{\partial \mathbf{p}^1} \right)^{-1} \frac{1}{n} \mathbf{1}_s,$$

which is equal to $\frac{1}{n(n-1)} \mathbf{1}_s$ by Lemma 1. This implies positive incentives to increase prices on every side locally.²⁰

Second, we identify the magnitudes of the cost reductions, due to post-merger synergies, to make the equilibrium prices unaffected. This result can be useful when all the pre-merger prices are positive.

Proposition 7. *Suppose $n > 2$. There exists a new cost vector $\hat{\mathbf{c}} = (\hat{c}_1, \dots, \hat{c}_s)$ for the merging platforms 1 and 2 such that the equilibrium prices after the merger under the new cost structure $(\hat{\mathbf{c}}, \hat{\mathbf{c}}, \mathbf{c}, \dots, \mathbf{c})$ stay the same as in Theorem 1 with original symmetric cost structure $(\mathbf{c}, \mathbf{c}, \mathbf{c}, \dots, \mathbf{c})$. More precisely, we have*

$$\hat{\mathbf{c}} = \mathbf{c} - \frac{1}{n-2} (\mathbf{p}^* - \mathbf{c})$$

where the symmetric pricing \mathbf{p}^* is given by Theorem 1. The percentage of profit increase by each merging platform from the above cost saving equals $1/(n-2)$.

Proposition 7 suggests that to keep the equilibrium prices unchanged, the cost reduction on each side by the merger needs to be proportional to the pre-merger margin (when the margin is positive), and this proportion depends on n but is not affected by side i .

Moreover, post-merger equilibrium prices and equilibrium market allocations stay the same as before, each customer obtains the same expected surplus. The benefit to each merging platform from the above cost saving is proportional to equilibrium profit as the cost reduction needed is proportional to the pre-merger margin.

6 Discriminatory vs uniform pricing

The equilibrium prices in Theorem 1 in general exhibit some degree of price discrimination across different sides. Now suppose government regulation prohibits this type

²⁰Note that if UPP vector is defined by $-\left(dg \frac{\partial \mathbf{x}^1}{\partial \mathbf{p}^1}\right)^{-1} \left(\frac{\partial \mathbf{x}^2}{\partial \mathbf{p}^1}\right)' (\mathbf{p}^2 - \mathbf{c})$ as in Affeldt et al. (2013), where $dg\mathbf{A}$ denotes the diagonal matrix which consists of the diagonal elements in \mathbf{A} , then the above argument suggests that in our setting $UPP = \frac{1}{n(n-1)} (dg\mathbf{E}) \mathbf{1}_s > \mathbf{0}$, and hence this modified measure of UPP provides an indication of the merged platforms to raise their prices locally.

of pricing behavior by forcing each platform to charge a uniform price on all sides. How does this policy affect the equilibrium prices, profits of platforms, and consumer welfare?

The following theorem determines the symmetric equilibrium uniform price:

Theorem 2. *Suppose $c_i = c, \forall i \in \mathcal{S}$ and all the platforms adopt the uniform pricing rule. Then there exists a symmetric equilibrium uniform price p^u with*

$$p^u = c + \frac{s/n}{\sum_{i,j} E_{ij}}, \quad (15)$$

where \mathbf{E} is given by (14).

Here \mathbf{E} is the semi-elasticity matrix $\partial \mathbf{x}^1 / \partial \mathbf{p}^1$ at the equilibrium prices. The pricing formula in (15) follows from standard optimal pricing rule, where s/n is the aggregate demand in equilibrium, and $\sum_{i,j} E_{ij}$ is the aggregate marginal demand increment when a platform raises his prices uniformly on all sides. Notice that there should be still product differentiation effect and subsidy effect, but the two effects cannot be isolated as in the case of discriminatory pricing. Nevertheless, the mark-up in (15) is explicitly determined by the primitives of the model.

Under both pricing rules, the demand profiles are identical, hence the total surplus remains unchanged. However, a ban on price discrimination makes some group(s) better off and other group(s) worse off, and it also affects the platform profits. To compare platform profits and consumer surplus between the two pricing rules, it suffices to compare the average prices across sides, as shown in the following Proposition.

Proposition 8. *The equilibrium outcomes under discriminatory pricing and uniform pricing characterized in Theorem 1 and 2 have the following properties:*

- (i) $\Pi^d > \Pi^u$ if and only if $\sum_i p_i^* / s > p^u$;
- (ii) For $i \in \mathcal{S}$, $CS_i^d > CS_i^u$ if and only if $p_i^* < p^u$;
- (iii) $TS^d = TS^u$.

Since the equilibrium prices in Theorem 1 and Theorem 2 have explicit expressions, the sign of $\sum p_i^* / s - p^u$ can be determined. Let us first consider the benchmark case without externalities.

Proposition 9. *Suppose that at least two of $M_i, i \in \mathcal{S}$ differ. Without externalities, $\sum_i p_i^* / s > p^u$ and the platforms strictly prefer discriminatory pricing over uniform pricing.*

The logic for the superiority of discriminatory pricing over uniform pricing is as follows: *Without externalities*, we have

$$p^u - c = \frac{s/n}{\sum_{i \in \mathcal{S}} h_i(0; n)} = \left[\frac{1}{s} \sum_{i \in \mathcal{S}} M_i^{-1} \right]^{-1},$$

which is the harmonic mean of the product differentiation effects $M_i, i \in \mathcal{S}$. Meanwhile,

$$\sum_{i \in \mathcal{S}} (p_i^* - c)/s = \frac{1}{s} \sum_{i \in \mathcal{S}} M_i$$

which is the arithmetic mean. Since the arithmetic mean is greater than or equal to the harmonic mean, $p^u \leq \sum_{i \in \mathcal{S}} p_i^*/s$, where the inequality must be strict by the assumption on $M_i, i \in \mathcal{S}$. By Proposition 8, $\Pi^d > \Pi^u$, i.e., platforms obtain strictly higher profits under discriminatory pricing. By continuity, the above results hold when the degree of externalities is small.

The following Proposition provides a comparison between the average price under discrimination and the uniform price in the presence of cross-side externalities when n is large. Let $M_i(\infty) = \lim_{n \rightarrow +\infty} M_i(n)$ denote the product differentiation effect in the limit.²¹

Proposition 10. *Suppose that the assumptions in Proposition 2 and 3 hold, and that at least two $M_i(\infty), i \in \mathcal{S}$, differ. Then $\lim_{n \rightarrow +\infty} \sum_{i \in \mathcal{S}} p_i^*/s > \lim_{n \rightarrow +\infty} p^u$.*

In the presence of externalities across sides, if the externality functions satisfy the assumptions in Proposition 3 then the impact of cross subsidies $\eta_i, i \in \mathcal{S}$ decreases as competition increases and vanishes in the limit. The price comparison is then largely determined by the product differentiation effects, and hence in the limit we have similar results to that in absence of externalities. Proposition 10 suggests that there exists a cutoff \bar{n} such that platforms prefer price discrimination over uniform pricing as long as $n > \bar{n}$. Moreover, if at least one $M_i(\infty)$ is zero, $\lim_{n \rightarrow \infty} p^u$ equals c ; if at least one $M_i(\infty)$ is nonzero, $\lim_{n \rightarrow \infty} \sum_{i \in \mathcal{S}} p_i^*/s$ is strictly above c .

The above result, although applicable only for large n , is valid for any s . For two-sided markets, we have the following simpler characterization for any finite n .

²¹Under the assumptions specified by Proposition 2, the product differentiation term $M_i(n)$ is monotonically decreasing in n , and hence $\lim_{n \rightarrow +\infty} M_i$ must exist.

Corollary 2. *Suppose $s = 2$ and the assumptions in Proposition 3 hold. Then $\Pi^d > \Pi^u$ and $(p_1^* + p_2^*)/2 > p^u$ if and only if*

$$|M_1(n) - M_2(n) - \frac{(\gamma_{11} - \gamma_{22})\rho'(\frac{1}{n})}{n-1}| > \frac{1}{n-1}|(\gamma_{12} - \gamma_{21})\rho'(\frac{1}{n})|. \quad (16)$$

In other words, for any finite number n , price discrimination leads to higher profits for platforms and higher average prices in two-sided markets whenever the difference in product differentiation effects between the two groups is significant, relatively to the difference in own and cross-side externalities between the two groups.

The condition in (16) becomes much simpler in the case of linear forms of externalities with $\gamma_{11} = \gamma_{22} = 0$ (i.e., no within-side externalities), $\gamma_{12} = \alpha_1$, and $\gamma_{21} = \alpha_2$. For $n = 2$ with the Hotelling specification as in Example 1, Corollary 2 states that

$$\Pi^d > \Pi^u \text{ if and only if } |t_1 - t_2| > |\alpha_1 - \alpha_2|,$$

which is the same as the condition identified by Armstrong (2006) (see equation (15) in his paper). Our result in Corollary 2 extends his analysis to allow for a more general model of product differentiation with an arbitrary number of competitors. For Gumbel distributions discussed in Proposition 5, Corollary 2 implies that

$$\Pi^d > \Pi^u \text{ if and only if } n|\beta_1 - \beta_2| > |\alpha_1 - \alpha_2|.$$

Due to cross-side externalities, it is likely that a small number of platforms prefer uniform pricing over discriminatory pricing. However, when $\beta_1 \neq \beta_2$, eventually $\Pi^d > \Pi^u$ for large n . As competition increases, platforms are more likely to engage in price discrimination.

7 Outside options

In this section, we study the impacts of customers' outside options on equilibrium prices. For simplicity, we focus on two-sided market, Gumbel distributions with parameters β_1, β_2 on the two sides, and a linear form of cross-side externalities with parameters α_1 and α_2 .

We assume explicitly that on each side i , there is an outside option which offers customers utility $u_i^0 = v_i^0 + \epsilon_i^0$, $i = 1, 2$. Without loss of generality, we normalize intrinsic values $v_i = 0$, $i = 1, 2$. Under these assumptions, the demands on both sides are determined by the following system of nonlinear equations:²²

²²The existence is similar to Proposition 1. The uniqueness holds when $\max(\alpha_1, \alpha_2) < 2 \min(\beta_1, \beta_2)$.

$$\begin{cases} x_1^k = \frac{e^{(\alpha_1 x_2^k - p_1^k)/\beta_1}}{\sum_{j=1}^n e^{(\alpha_1 x_2^j - p_1^j)/\beta_1} + e^{v_1^0/\beta_1}} \\ x_2^k = \frac{e^{(\alpha_2 x_1^k - p_2^k)/\beta_2}}{\sum_{j=1}^n e^{(\alpha_2 x_1^j - p_2^j)/\beta_2} + e^{v_2^0/\beta_2}} \end{cases} \quad k = 1, 2, \dots, n. \quad (17)$$

Let $\mathbf{x}^0 = (x_1^0, x_2^0) = (1 - \sum_{k=1}^n x_1^k, 1 - \sum_{k=1}^n x_2^k)$ denote demand vector of the outside options.

To determine the equilibrium prices, we need to compute the Jacobian matrix of each platform's demand with respect to his own prices. Due to cross-side externalities, the demand systems are implicitly determined by (17). As such the Jacobian matrices for the platforms are interdependent and quite complicated. With symmetry, we are able to simplify these matrices by using the following notation, and especially by examining how the difference in demands between two platforms change with respect to one platform's price increase.

Lemma 2. *With outside options, at any symmetric price $\mathbf{p} = (p_1, p_2)$ with symmetric allocation $\mathbf{x} = (x_1, x_2)$, the following conditions hold: for any $k, k' \in \mathcal{N}$ and $k' \neq k$,*

$$-\frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k} = \frac{n-1}{n} \mathbf{Y} + \frac{1}{n} \mathbf{W} \quad \text{and} \quad \frac{\partial \mathbf{x}^{k'}}{\partial \mathbf{p}^k} = \frac{1}{n} (\mathbf{Y} - \mathbf{W}),$$

where

$$\mathbf{Y}^{-1} = \begin{bmatrix} \frac{\beta_1}{x_1} & -\alpha_1 \\ -\alpha_2 & \frac{\beta_2}{x_2} \end{bmatrix} \quad \text{and} \quad \mathbf{W}^{-1} = \begin{bmatrix} \frac{\beta_1}{x_1(1-nx_1)} & -\alpha_1 \\ -\alpha_2 & \frac{\beta_2}{x_2(1-nx_2)} \end{bmatrix}. \quad (18)$$

At any symmetric price $\mathbf{p}^k = \mathbf{p} = (p_1, p_2)$ with symmetric allocation $\mathbf{x}^k = \mathbf{x} = (x_1, x_2)$, define $\mathbf{Y} := \frac{\partial(\mathbf{x}^{k'} - \mathbf{x}^k)}{\partial \mathbf{p}^k}$ for any two different platform k and k' , and $\mathbf{W} := \frac{\partial \mathbf{x}^0}{\partial \mathbf{p}^k}$ for any k . Here \mathbf{W} represents the proportion of the customers who switch from a platform to the outside option when the platform raises his prices. Similarly, \mathbf{Y} describes how the difference in demands between two platforms, k' and k , change when platform k raises its prices. Moreover, since the sum of all the shares equals 1, it follows that it follows that for any k , $-\frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k} = \frac{n-1}{n} \mathbf{Y} + \frac{1}{n} \mathbf{W}$, and $\frac{\partial \mathbf{x}^{k'}}{\partial \mathbf{p}^k} = \frac{1}{n} (\mathbf{Y} - \mathbf{W})$. In other words, the Jacobian matrices of each platform's demand with respect to his own prices and any other platform's prices are completely determined by \mathbf{Y} and \mathbf{W} .

Lemma 2 extends Lemma 1 to the case with outside options. In this case the symmetric allocation \mathbf{x} is not equal, but strictly less than $\frac{1}{n} \mathbf{s}_s$. The exact values of \mathbf{x} depends implicitly on the price \mathbf{p} , the degree of externalities, the parameters of the distributions,

and the relative attractiveness of outside options. As a consequence, the above Jacobian matrices do not have the independence on the symmetric price \mathbf{p} as that in Lemma 1. Nevertheless, both \mathbf{Y} and \mathbf{W} take simple forms under Gumbel distributions.

To understand the logic of Lemma 2, we consider the relative market shares between platform k and k' on both sides:

$$\begin{cases} \beta_1 \ln \frac{x_1^{k'}}{x_1^k} = \alpha_1(x_2^{k'} - x_2^k) + (p_1^k - p_1^{k'}) \\ \beta_2 \ln \frac{x_2^{k'}}{x_2^k} = \alpha_2(x_1^{k'} - x_1^k) + (p_2^k - p_2^{k'}). \end{cases} \quad (19)$$

Differentiating the above system with respect to $\mathbf{p}^k = (p_1^k, p_2^k)$ and using the symmetry across platforms, $x_i^k = x_i^{k'} = x_i, i = 1, 2$, yields

$$\begin{bmatrix} \frac{\beta_1}{x_1} & 0 \\ 0 & \frac{\beta_2}{x_2} \end{bmatrix} \frac{\partial(\mathbf{x}^{k'} - \mathbf{x}^k)}{\partial \mathbf{p}^k} = \begin{bmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{bmatrix} \frac{\partial(\mathbf{x}^{k'} - \mathbf{x}^k)}{\partial \mathbf{p}^k} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which leads to the expression for $\mathbf{Y} = \frac{\partial(\mathbf{x}^{k'} - \mathbf{x}^k)}{\partial \mathbf{p}^k}$ in Lemma 2. Following a similar approach, we derive $\mathbf{W} = \frac{\partial \mathbf{x}^0}{\partial \mathbf{p}^k}$ by differentiating the demands of outside options \mathbf{x}^0 with respect to \mathbf{p}^k . The above technique gives clean expressions for the Jacobian matrices or demand elasticities by utilizing the symmetry across platforms and the features of the multinomial logit demand systems.²³

We are interested in a symmetric equilibrium in which each platform charges price $\mathbf{p}^i = \mathbf{p}^* = (p_1^*, p_2^*)$ with demand $\mathbf{x}^i = \mathbf{x}^* = (x_1^*, x_2^*)$, where (17) simplifies to

$$\begin{cases} p_1^* = -v_1^0 + \beta_1 \ln \frac{1-nx_1^*}{x_1^*} + \alpha_1 x_2^* \\ p_2^* = -v_2^0 + \beta_2 \ln \frac{1-nx_2^*}{x_2^*} + \alpha_2 x_1^*. \end{cases} \quad (20)$$

Using Lemma 1, we provide the following characterization of the symmetric equilibrium.²⁴

Theorem 3. *Suppose $s = 2$, Gumbel distributions, linear externalities, and outside options. Then the symmetric equilibrium $(\mathbf{p}^*, \mathbf{x}^*)$ is characterized by*

$$\mathbf{p}^* = \mathbf{c} + \left[\frac{n-1}{n} \mathbf{Y} + \frac{1}{n} \mathbf{W} \right]^{-1'} \mathbf{x}^* \quad (21)$$

²³In general, at any asymmetric price profile, the computation of elasticity matrix involves implicit differentiating (17) and inverting a $2n \times 2n$ matrix.

²⁴Theorem 3 and Lemma 2 can be easily extended to any s (including $s = 1$) and general forms of externalities (not just linear form). Our proof utilizes the IIA property of the IID Gumbel distributions but our characterization is similar for general IID shocks even though the proof needs to be modified.

together with (20), where \mathbf{Y} and \mathbf{W} are given in Lemma 2.

Theorem 3 provides a characterization of the symmetric equilibrium prices p_1^*, p_2^* and market shares x_1^*, x_2^* which are determined by four nonlinear equations (20) and (21). When outside options are explicitly available, the equilibrium prices do not have a simple formula as compared with the ones in Theorem 1. The following Corollary summarizes the impacts of outside options.

Corollary 3. *Assume $n \geq 2$, the following statements are equivalent:²⁵*

- (1) *There are no outside options ($v_1^0 = v_2^0 = -\infty$);*
- (2) *Markets are fully covered;*
- (3) $\mathbf{W} = \mathbf{0}$.

Under either condition above, the separation property holds:

$$\mathbf{p}^* - \mathbf{c} = -\left[\frac{n-1}{n}\mathbf{Y}\right]^{-1'} \cdot \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \frac{n}{n-1}\beta_1 - \frac{1}{n-1}\alpha_2 \\ \frac{n}{n-1}\beta_2 - \frac{1}{n-1}\alpha_1 \end{bmatrix}.$$

Corollary 3 suggests that under one of the equivalent conditions, the equilibrium prices satisfy a separation property in the sense that the product differentiation effect and the subsidy effect can be separated and determined by the parameters of the model and that the price-cost margins can be decomposed additively into these two effects. When there are effective outside options (i.e., the intrinsic values of the outside options are finite), the above separation property does not hold, and the equilibrium prices and shares are implicitly determined by (20) and (21). Consequently, the product differentiation effect on each side cannot be isolated from the subsidy effect on the same side, and moreover, the equilibrium price on side 1 depends also on the product differentiation effect on side 2, β_2 and the marginal externalities that side 2 imposed on side 1, α_1 .

Nevertheless, the following property generally holds.

Corollary 4. *For $i = 1, 2$, $p_i^* \rightarrow c_i + \beta_i$, as $n \rightarrow +\infty$.*

As the number of platforms increases unboundedly, the impact of outside options is negligibly small, and therefore with or without outside options the equilibrium prices converge to the same limit.

²⁵When $n = 1$, Theorem 3 still holds, and (20) and (21) involve a system of nonlinear equations.

8 Conclusion

In this paper, we have analyzed a model of price competition among differentiated platforms in multi-sided markets. In specifying customers' preferences, we have incorporated a general functional form of externalities (both within and cross sides) into a discrete choice framework with general distributions of random utilities. Based on symmetry across platforms and full market coverage, we have explicitly provided sufficient conditions for the existence and uniqueness of the equilibrium and found that the symmetric equilibrium prices take simple, decomposable formulae. The simplicity of the equilibrium pricing formulae allows us to conduct several interesting comparative statics with respect to platform competition. By explicitly considering outside options, we have illustrated the importance of full market coverage, along with symmetry and single-homing assumption, in deriving the simple pricing formulae.

In our analysis we have assumed one-stopping shopping (i.e., single-homing) for customers on all sides and thus ignored the possibility of multi-homing, an important issue in studying multi-sided markets. Here, we want to note that our baseline model can be adapted to allow for multi-homing by certain side(s). Following [Armstrong \(2006\)](#), one way of modelling multi-homing in our setting is to assume that the customers on the multi-homing side decide to join platform k , independently of her decision to join k' or not, as long as her utility from joining k exceeds that from an outside option.²⁶ Using our techniques in [Section 7](#), we could characterize the symmetric equilibrium in two-sided markets, although the Jacobian matrices would take slightly different forms. The explicit expressions for the equilibrium prices are not available, due to the presence of outside options. In general, we expect that the market share and price on the multi-homing side would be higher while the platform would give a larger subsidy to the customers on the single-homing side, as compared with the single-homing benchmark case. This can be seen in the limiting case as competition becomes sufficiently large: In contrast with the case of single-homing on both sides, the demand on the multi-homing side does not approach zero, and the price on the multi-homing side is strictly above cost, while the platforms still subsidize the single-homing side although the market share on that side decreases to 0. Further analysis along this line is left for future research.

²⁶Admittedly, there are alternative ways of modeling payoffs from multi-homing. See, for instance, [Armstrong and Wright \(2007\)](#); [Jullien and Pavan \(2016\)](#).

Appendix

A Proofs

Proof of Proposition 1: Given \mathbf{P} , the right hand side of (3) defines a mapping Σ from Ω to Ω , where

$$\Omega = \{(x_i^k, i \in \mathcal{S}, k \in \mathcal{N}) \in \mathbf{R}^{sn} | x_i^k \geq 0, \forall i, k; \text{ and } \sum_{k \in \mathcal{N}} x_i^k = 1, \forall i \in \mathcal{S}\}.$$

Clearly, Ω is compact and convex, and Σ is continuous on Ω . Applying *Brouwer's fixed point theorem*, we obtain at least one fixed point of Σ , which is a PE in the participation stage. \square

Proof of Theorem 1: To support the price profile $\mathbf{P}^* = (\mathbf{p}^*, \dots, \mathbf{p}^*)$ in (5) and $\mathbf{x}^* = \frac{1}{n}\mathbf{1}_s$ as the subgame perfect Nash equilibrium outcome, we need to specify a PE for any price profile, which can be described as follows:

- (i) On the equilibrium path where every platform chooses \mathbf{p}^* , pick the PE that gives each platform $\mathbf{x}^* = \frac{1}{n}\mathbf{1}_s$.
- (ii) If only one platform, say platform 1, deviates from \mathbf{p}^* to \mathbf{p}^1 , we consider the following *semi-symmetric* demand profile

$$\left(\mathbf{q}^1, \frac{\mathbf{1}_s - \mathbf{q}^1}{n-1}, \dots, \frac{\mathbf{1}_s - \mathbf{q}^1}{n-1} \right),$$

which is a PE at $(\mathbf{p}^1, \mathbf{p}^*, \dots, \mathbf{p}^*)$ if and only if \mathbf{q}^1 is any solution to the following system of equations (it always exists by *Brouwer's fixed point theorem*):

$$\begin{aligned} q_i^1 &= \Pr(v_i + \phi_i(\mathbf{q}^1) - p_i^1 + \epsilon_i^1 \geq \max_{t \neq 1} \{v_i + \phi_i(\overbrace{\mathbf{q}^t}^{=\frac{\mathbf{1}_s - \mathbf{q}^1}{n-1}}) - \overbrace{p_i^t}^{:=p_i^*} + \epsilon_i^t\}) \\ &= \Pr(\epsilon_i^1 - \max(\epsilon_i^2, \dots, \epsilon_i^n) \geq p_i^1 - p_i^* - \phi_i(\mathbf{q}^1) + \phi_i(\frac{\mathbf{1}_s - \mathbf{q}^1}{n-1})) \\ &= 1 - H_i\left(p_i^1 - p_i^* - \phi_i(\mathbf{q}^1) + \phi_i(\frac{\mathbf{1}_s - \mathbf{q}^1}{n-1}); n\right), i = 1, 2, \dots, s. \end{aligned} \quad (22)$$

To be consistent with item (i) above, if \mathbf{p}^1 happens to be exactly \mathbf{p}^* , we pick $\mathbf{q}^1 = \frac{1}{n}\mathbf{1}_s$, which solves (22) by symmetry.

(iii) For any \mathbf{P} involving more than 2 platform deviating from \mathbf{p}^* , pick any PE at \mathbf{P} , which always exists by Proposition 1.

We now check that there is no profitable deviation by any platform from the proposed equilibrium candidate. Note that the deviating profit of platform 1 at price \mathbf{p}^1 is

$$\Pi^1(\mathbf{p}^1) = \langle \mathbf{p}^1 - \mathbf{c}, \mathbf{q}^1(\mathbf{p}^1) \rangle$$

where $\mathbf{q}^1(\mathbf{p}^1)$ is a solution to (22). By inverting (22), we observe that \mathbf{p}^1 must satisfy the following inverse demand system:

$$p_i^1(\mathbf{q}^1) = H_i^{-1}(1 - q_i^1; n) + p_i^* + \phi_i(\mathbf{q}^1) - \phi_i\left(\frac{\mathbf{1}_s - \mathbf{q}^1}{n-1}\right), i = 1, 2, \dots, s. \quad (23)$$

at quantity $\mathbf{q}^1(\mathbf{p}^1)$. Thus, changing variables, we can rewrite the profit of platform 1 as follows:

$$\Pi^1(\mathbf{p}^1) = \sum q_i^1 \left(H_i^{-1}(1 - q_i^1; n) + p_i^* + \phi_i(\mathbf{q}^1) - \phi_i\left(\frac{\mathbf{1}_s - \mathbf{q}^1}{n-1}\right) - c_i \right) = R(\mathbf{q}^1(\mathbf{p}^1); \mathbf{p}^* - \mathbf{c})$$

by the definition of R .

Notice that $R(\frac{1}{n}\mathbf{1}_s; \mathbf{p}^* - \mathbf{c}) = 1/n \sum_{i \in \mathcal{S}} (p_i^* - c_i)$ is just the profit of platform 1 on the equilibrium path. Deviating from \mathbf{p}^* to \mathbf{p}^1 is not profitable if $\frac{1}{n}\mathbf{1}_s$ maximizes $R(\mathbf{q}^1; \mathbf{p}^* - \mathbf{c})$ over $\mathbf{q}^1 \in [0, 1]^s$, which is indeed true by the following two observations: At $\frac{1}{n}\mathbf{1}_s$, the first-order conditions are satisfied, i.e.,

$$\frac{\partial R(\mathbf{q}^1; \mathbf{p}^* - \mathbf{c})}{\partial q_i^1} \Big|_{\mathbf{q}^1 = \frac{1}{n}\mathbf{1}_s} = \frac{1}{n} \frac{-1}{h_i(0; n)} + \frac{1}{n} \sum_{j \in \mathcal{S}} \left(1 + \frac{1}{n-1}\right) \frac{\partial \phi_j(\mathbf{z})}{\partial z_i} \Big|_{\mathbf{z} = \frac{1}{n}\mathbf{1}_s} + p_i^* - c = 0$$

by the price \mathbf{p}^* given by (5) in Theorem 1. Here we have used the following identities:

$$H_i(0; n) = 1 - \frac{1}{n}, \quad H_i^{-1}\left(1 - \frac{1}{n}; n\right) = 0, \quad \left(\phi_i(\mathbf{q}^1) - \phi_i\left(\frac{\mathbf{1}_s - \mathbf{q}^1}{n-1}\right) \right) \Big|_{\mathbf{q}^1 = \frac{1}{n}\mathbf{1}_s} = 0, \forall i.$$

Moreover, $R(\mathbf{q}^1; \mathbf{p}^* - \mathbf{c})$ is globally concave in \mathbf{q}^1 by Assumption 1. The claim follows. \square

Proof of Proposition 2: We first illustrate the proof by consider the IID case. We utilize the following identity (by Zhou (2017)):

$$\frac{h_i(0; n)}{1 - H_i(0; n)} = \int \frac{f_i(\theta)}{1 - F_i(\theta)} dF_i^{2:n}(\theta)$$

where $F_i^{2:n}(\cdot)$ denotes the CDF of the second highest order statistics of n IID draws from F_i . Clearly, the right-hand-side is increasing in n under the log-concavity of $1 - F_i(\cdot)$.

For the CIID case, the proof follows from a generalization of the above identity:

$$\frac{h_i(0; n)}{1 - H_i(0; n)} = \int \left\{ \int \frac{f_i(\theta|\tau_i)}{1 - F_i(\theta|\tau_i)} dF_i^{2:n}(\theta|\tau_i) \right\} dG_i(\tau_i)$$

Since the bracket term for each fixed τ_i on the right-hand-side is increasing in n , by integration, the right-hand-side is also increasing in n . The claim follows. \square

Proof of Proposition 3: Note that $\frac{\rho'(1/n)}{n-1} = \frac{1}{n}\rho'(\frac{1}{n})\frac{n}{n-1}$. Given that $0 < x\rho''(x) + \rho'(x) = (x'\rho(x))'$, the term $\frac{1}{n}\rho'(\frac{1}{n})$ decreases in n , moreover $\frac{n}{n-1}$ clearly decreases in n , as a result, $\frac{\rho'(1/n)}{n-1}$ decreases in n . Since $\lim_{x \rightarrow 0} x\rho'(x) = 0$, $\lim_{n \rightarrow \infty} \frac{\rho'(1/n)}{n-1} = 0$, therefore $\eta_i(n) \rightarrow 0$ as $n \rightarrow \infty$. \square

Proof of Proposition 4: Given the equilibrium prices and volumes in Theorem 1, the equilibrium profit for any platform is

$$\Pi(n) = \langle \frac{1}{n}\mathbf{1}_s, \mathbf{p}^* - \mathbf{c} \rangle = \frac{1}{n} \sum_{i \in \mathcal{S}} (M_i - \eta_i).$$

For the customer on side i , the expected consumer surplus is

$$CS_i(n) = \mathbb{E}[\max_{k=1,2,\dots,n} \{v_i + \phi_i(\frac{1}{n}\mathbf{1}_s) - p_i^* + \epsilon_i^k\}] = v_i - c_i - M_i + \eta_i + \phi_i(\frac{1}{n}\mathbf{1}_s) + \delta_i.$$

The total surplus is

$$TS(n) = \sum_{i \in \mathcal{S}} CS_i + n\Pi = \sum_{i \in \mathcal{S}} \left(v_i - c_i + \phi_i(\frac{1}{n}\mathbf{1}_s) + \delta_i \right).$$

\square

Proof of Corollary 1: Under the given assumption $\frac{\partial TS}{\partial n} - \frac{n-1}{n}\Pi = \sum_{i \in \mathcal{S}} (\frac{\partial \delta_i}{\partial n} - \frac{n-1}{n^2}M_i) \leq 0$ for any n . Let n^{FE} denote the number of platform with free entry. Then it must be the case that $\Pi(n^{FE}) = K$. As a result, $\frac{\partial TS(n^{FE})}{\partial n} - K \leq \frac{n^{FE}-1}{n^{FE}}\Pi(n^{FE}) - K = \frac{n^{FE}-1}{n^{FE}}K - K = -\frac{K}{n^{FE}} < 0$. The result just follows. \square

Proof of Proposition 5: The cumulative distribution function of the Gumbel distribution (Type I extreme value distribution) with parameter β_i is $e^{-e^{-x/\beta_i}}$. For IID Gumbel

distribution with parameter β_i , the follow holds(see [Anderson et al. \(1992\)](#))

$$H_i(\theta; n) = \frac{n-1}{e^{-\theta/\beta_i} + (n-1)}, \text{ and } M_i = \frac{1 - H_i(0; n)}{h_i(0; n)} = \frac{n}{n-1}\beta_i \quad (24)$$

Moreover, it can be shown that

$$\delta_i = \mathbb{E}[\max_{k=1,2,\dots,n} \epsilon_i^k] = \beta_i(\ln(n) + \kappa).$$

Assume linear externalities, Theorem 1 implies

$$p_i^*(n) = c_i + \frac{n}{n-1}\beta_i - \frac{1}{n-1}\bar{\gamma}_i, \quad i \in \mathcal{S}.$$

as given in part (i) of Proposition 5. Moreover $\frac{\partial p_i^*}{\partial n} = -(\beta_i - \bar{\gamma}_i)/(n-1)^2$, so p_i^* decreases with n if and only if $\beta_i > \bar{\gamma}_i$. For equilibrium profit, $\Pi(n) = \frac{1}{n} \sum_{i \in \mathcal{S}} (p_i^* - c_i) = \frac{1}{n} \sum_{i \in \mathcal{S}} (\frac{n}{n-1}\beta_i - \frac{1}{n-1}\bar{\gamma}_i) = \frac{n\bar{\beta} - \bar{\gamma}}{n(n-1)}$, which proves (10). The monotonicity result just follows The expressions of total surplus in part (iii) directly follow from Proposition 4 and equation (24). Total surplus is increasing in n as $\frac{\partial TS(n)}{\partial n} = \frac{n-1}{n}\Pi(n) > 0$. Part (iv) directly follows from Corollary 1 as for Gumbel distribution, by equation (24), $\frac{\partial \delta_i}{\partial n} - \frac{n-1}{n^2}M_i = \frac{\beta_i}{n} - \frac{n-1}{n^2} \frac{n}{n-1}\beta_i = 0$. \square

Proof of Lemma 1: We present a general version of Lemma 1.

Lemma 1': *With full market coverage, suppose platform k charges prices \mathbf{p}^k with demand \mathbf{x}^k while other platforms choose symmetric prices \mathbf{p} with the same demand $\frac{\mathbf{1}_s - \mathbf{x}^k}{n-1}$, the following hold: for any $k' \in \mathcal{N}$ and $k' \neq k$,*

$$\frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k} = -\tilde{\mathbf{E}}, \quad \text{and} \quad \frac{\partial \mathbf{x}^{k'}}{\partial \mathbf{p}^k} = \frac{1}{n-1}\tilde{\mathbf{E}},$$

where

$$\tilde{\mathbf{E}} = \left\{ \text{Diag}\left(\frac{1}{h_1(\theta_1; n)}, \dots, \frac{1}{h_s(\theta_s; n)}\right) - \left(\Psi(\mathbf{x}^k) + \frac{1}{n-1}\Psi\left(\frac{\mathbf{1}_s - \mathbf{x}^k}{n-1}\right)\right) \right\}^{-1}$$

and $\theta_i = p_i^k - p_i - \phi_i(\mathbf{x}^k) + \phi_i\left(\frac{\mathbf{1}_s - \mathbf{x}^k}{n-1}\right), i \in \mathcal{S}$.

Clearly Lemma 1 is a special case of Lemma 1' under the condition that $\mathbf{p}^k = \mathbf{p}$, $\mathbf{x}^k = \frac{1}{n}\mathbf{1}_s$, and hence $\theta_i = 0, \forall i$. In this case, $\tilde{\mathbf{E}}$ simplifies to (14) in Lemma 1.

Proof of Lemma 1': At the price profile $\mathbf{P} = (\mathbf{p}^k, \mathbf{p}, \dots, \mathbf{p})$, the demand profile $(\mathbf{x}^k, \frac{\mathbf{1}_s - \mathbf{x}^k}{n-1}, \dots, \frac{\mathbf{1}_s - \mathbf{x}^k}{n-1})$ is a participation equilibrium, by the same logic in the proof of Theorem 1, when the following condition holds

$$x_i^k = 1 - H_i \left(p_i^k - p_i - \phi_i(\mathbf{x}^k) + \phi_i\left(\frac{\mathbf{1}_s - \mathbf{x}^k}{n-1}\right); n \right), i = 1, 2, \dots, s.$$

Differentiate the above system with respect to \mathbf{p}^k yields

$$\frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k} = -\text{Diag}(h_1(\theta_1; n), \dots, h_s(\theta_s; n)) \left[\mathbf{I}_s - \left(\Psi(\mathbf{x}^k) + \frac{1}{n-1} \Psi\left(\frac{\mathbf{1}_s - \mathbf{x}^k}{n-1}\right) \right) \frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k} \right].$$

where $\theta_i = p_i^k - p_i - \phi_i(\mathbf{x}^k) + \phi_i\left(\frac{\mathbf{1}_s - \mathbf{x}^k}{n-1}\right), i \in \mathcal{S}$.

Solving for $\frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k}$ yields

$$\frac{\partial \mathbf{x}^k}{\partial \mathbf{p}^k} = - \left\{ \text{Diag}\left(\frac{1}{h_1(\theta_1; n)}, \dots, \frac{1}{h_s(\theta_s; n)}\right) - \left(\Psi(\mathbf{x}^k) + \frac{1}{n-1} \Psi\left(\frac{\mathbf{1}_s - \mathbf{x}^k}{n-1}\right) \right) \right\}^{-1}.$$

Moreover, we know that $\frac{\partial \mathbf{x}^{k'}}{\partial \mathbf{p}^k} = -\frac{1}{n-1} \frac{\partial \mathbf{x}^{k'}}{\partial \mathbf{p}^k}$. Lemma 1' just follows. \square

Proof of Proposition 6: Platform profit maximization at the symmetric equilibrium \mathbf{p}^* implies that

$$\left\{ \underbrace{\mathbf{x}^1}_{1/n\mathbf{1}_s} + \left\{ \frac{\partial \mathbf{x}^1}{\partial \mathbf{p}^1} \right\}' (\mathbf{p}^* - \mathbf{c}) \right\} \Big|_{\mathbf{p}^k = \mathbf{p}^*, \forall k \in \mathcal{N}} = \mathbf{0} \quad (25)$$

If platform 1 increases its prices \mathbf{p}^1 at the symmetric equilibrium, the marginal impact on its own profit is zero. However it will affect the profit of platform 2 by

$$\partial \Pi^2 / \partial \mathbf{p}^1 \Big|_{\mathbf{p}^k = \mathbf{p}^*, \forall k \in \mathcal{N}} = \left\{ \frac{\partial \mathbf{x}^2}{\partial \mathbf{p}^1} \right\}' (\mathbf{p}^* - \mathbf{c}) \Big|_{\mathbf{p}^k = \mathbf{p}^*, \forall k \in \mathcal{N}} = \frac{1}{(n-1)n} \mathbf{1}_s \succ \mathbf{0},$$

where the last step follows from the fact that $\frac{\partial \mathbf{x}^2}{\partial \mathbf{p}^1} \Big|_{\mathbf{p}^k = \mathbf{p}^*, \forall k \in \mathcal{N}} = -\frac{1}{n-1} \frac{\partial \mathbf{x}^1}{\partial \mathbf{p}^1} \Big|_{\mathbf{p}^k = \mathbf{p}^*, \forall k \in \mathcal{N}}$ (see Lemma 1) and (25). \square

Proof of Proposition 7: The merging platforms coordinate prices to maximize their joint profits:

$$\max_{\mathbf{p}^1, \mathbf{p}^2} \langle \mathbf{p}^1 - \hat{c}, \mathbf{x}^1 \rangle + \langle \mathbf{p}^2 - \hat{c}, \mathbf{x}^2 \rangle$$

The first-order condition with respect to \mathbf{p}^1 is

$$\mathbf{x}^1 + \sum_{l=1}^2 \left\{ \frac{\partial \mathbf{x}^l}{\partial \mathbf{p}^1} \right\}' (\mathbf{p}^l - \hat{\mathbf{c}}) = \mathbf{0}_s.$$

Since the equilibrium prices after merger stay the same as those before the merger \mathbf{p}^* , plugging $\mathbf{p}^k = \mathbf{p}^*$, $\mathbf{x}^k = \frac{1}{n} \mathbf{1}_s, \forall k$ into the above equation yields

$$\frac{1}{n} \mathbf{1}_s + \left\{ -\mathbf{E} + \frac{1}{n-1} \mathbf{E} \right\}' (\mathbf{p}^* - \hat{\mathbf{c}}) = \mathbf{0}_s,$$

where we use the fact that $\frac{\partial \mathbf{x}^1}{\partial \mathbf{p}^1} |_{\mathbf{P}=(\mathbf{p}^*, \dots, \mathbf{p}^*)} = -\mathbf{E}$ and $\frac{\partial \mathbf{x}^2}{\partial \mathbf{p}^1} |_{\mathbf{P}=(\mathbf{p}^*, \dots, \mathbf{p}^*)} = \frac{1}{(n-1)} \mathbf{E}$ by Lemma 1. As a consequence,

$$\mathbf{p}^* - \hat{\mathbf{c}} = \frac{1}{1 - 1/(n-1)} \mathbf{E}'^{-1} \frac{1}{n} \mathbf{1}_s = \frac{n-1}{n-2} (\mathbf{p}^* - \mathbf{c})$$

where the last step follows from (25). Hence, $\hat{\mathbf{c}} = \mathbf{c} - \frac{1}{n-2} (\mathbf{p}^* - \mathbf{c})$.

Since post-merger equilibrium prices and equilibrium market allocations stay the same as before, each customer obtains the same expected surplus. The benefit to each merging platform from the above cost saving equals

$$\Delta \Pi = \sum_{i=1}^s \frac{1}{n} (c_i - \hat{c}_i) = \sum_{i=1}^s \frac{1}{n} \frac{1}{n-2} (p_i^* - c_i) = \frac{1}{n-2} \Pi.$$

□

Proof of Theorem 2: Given that platform 2 to n charge the equilibrium price $\mathbf{p}^u = p^u \mathbf{1}_s$, platform 1's deviating profit from charging alternative uniform price p equals

$$\langle p \mathbf{1}_s - c \mathbf{1}_s, \mathbf{x}^1(p \mathbf{1}_s, \mathbf{p}^u, \dots, \mathbf{p}^u) \rangle$$

The first-order condition with respect to p , evaluated the symmetric uniform price p^u , is

$$\underbrace{\langle \mathbf{1}_s, \mathbf{x}^1(p \mathbf{1}_s, \mathbf{p}^u, \dots, \mathbf{p}^u) |_{p=p^u} \rangle}_{=\frac{1}{n} \mathbf{1}_s} + (p^u - c) \sum_{i,j \in \mathcal{S}} \left(\frac{\partial \mathbf{x}^1}{\partial \mathbf{p}^1} |_{\mathbf{p}^k = p^u \mathbf{1}_s, \forall k} \right)_{ij} = 0.$$

The rest of the proof follows from the following observation that at symmetric price $p^k = p^u \mathbf{1}_s, \forall k$ $\frac{\partial \mathbf{x}^1}{\partial \mathbf{p}^1} = -\mathbf{E}$ by Lemma 1. □

Proof of Proposition 8: Under both pricing rules, the equilibrium market share of each platform is the same, which is $\frac{1}{n}\mathbf{1}_s$. It follows that $\Pi^u = \frac{s}{n}(p^u - c)$, and $\Pi^d = \frac{1}{n}\sum_i(p_i^* - c)$. Thus, $\Pi^d > \Pi^u$ if and only if $\frac{1}{s}\sum_i p_i^* > p^u$. Moreover, $CS_i^d = v_i - c - p_i^* + \phi(\frac{1}{n}\mathbf{1}_s) + \delta_i$, and $CS_i^u = v_i - c - p^u + \phi(\frac{1}{n}\mathbf{1}_s) + \delta_i$. Thus, $CS_i^d > CS_i^u$ if and only if $p_i^* < p^u$. Clearly, the total surplus is not affected by the pricing rules, as the symmetric prices are just transfers between platforms and customers. \square

Proof of Proposition 9: It follows from the discussion in the text. \square

Proof of Proposition 10: By (14), we have

$$n\mathbf{E} = \left\{ \text{Diag}\{M_1(n), \dots, M_s(n)\} - \frac{1}{n-1}\Psi\left(\frac{1}{n}\mathbf{1}_s\right) \right\}^{-1}$$

Under the assumptions in Proposition 3, $\frac{1}{n-1}\Psi\left(\frac{1}{n}\mathbf{1}_s\right) = \frac{\rho'(1/n)}{n-1}\Gamma \rightarrow \mathbf{0}$ as $n \rightarrow \infty$. By Theorem 2, we have

$$\lim_{n \rightarrow +\infty} p^u - c = \left[\frac{1}{s} \sum_{i \in \mathcal{S}} \left(\lim_{n \rightarrow \infty} M_i(n) \right)^{-1} \right]^{-1}. \quad (26)$$

which is the harmonic mean of $\{M_i(\infty), i \in \mathcal{S}\}$. Similarly, under discriminatory pricing, we can show that $\lim_{n \rightarrow +\infty} p_i^* - c = \lim_{n \rightarrow +\infty} M_i(n) = M_i(\infty)$ as $\eta_i(n) \rightarrow 0$ by Proposition 3. Therefore

$$\lim_{n \rightarrow +\infty} \sum_{i \in \mathcal{S}} (p_i^* - c)/s = \frac{1}{s} \sum_{i \in \mathcal{S}} M_i(\infty), \quad (27)$$

which is the arithmetic mean of $\{M_i(\infty), i \in \mathcal{S}\}$. Since the harmonic mean is less or equal to the arithmetic mean, we must have $\lim_{n \rightarrow +\infty} p^u \leq \lim_{n \rightarrow +\infty} \sum_{i \in \mathcal{S}} p_i^*/s$. The strict inequality holds if and only if not all these $M_i(\infty)$ are the same.

Since the geometric mean is zero if at least one $M_i(\infty)$ is zero, in which case $\lim_{n \rightarrow +\infty} p^u - c = 0$; the arithmetic mean is positive if at least one $M_i(\infty)$ is positive, in which case $\lim_{n \rightarrow +\infty} \sum_{i \in \mathcal{S}} (p_i^* - c)/s > 0$. \square

Proof of Corollary 2: By Proposition 8, $\Pi^d > \Pi^u$ if and only if $p_1^* + p_2^* > 2p^u$ as $s = 2$. According to Theorem 1, we have

$$p_1^* = c + M_1(n) - \frac{n\rho'(1/n)}{n-1}(\gamma_{11} + \gamma_{21}), \text{ and } p_2^* = c + M_2(n) - \frac{n\rho'(1/n)}{n-1}(\gamma_{21} + \gamma_{22})$$

for discriminatory pricing, while by Theorem 2, the uniform price equals

$$p^u = c + \frac{2/n}{\sum E_{ij}},$$

where the matrix \mathbf{E} in this case reduces to:

$$\mathbf{E} = \left(\begin{bmatrix} \frac{1}{h_1(0;n)} & 0 \\ 0 & \frac{1}{h_2(0;n)} \end{bmatrix} - \frac{n\rho'(1/n)}{n-1} \mathbf{\Gamma} \right)^{-1} = \frac{1}{n} \left(\begin{bmatrix} M_1(n) & 0 \\ 0 & M_2(n) \end{bmatrix} - \frac{\rho'(1/n)}{n-1} \mathbf{\Gamma} \right)^{-1}$$

where $\mathbf{\Gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$. Plugging in these conditions and simplifying yield the condition as stated in Corollary 2. \square

Proof of Lemma 2: The formula for \mathbf{Y} follows from the logic in the main paper. For \mathbf{W} , we first differentiate the demand of the outside option \mathbf{x}^0

$$\begin{cases} x_1^0 = \frac{e^{v_1^0/\beta_1}}{\sum_{j=1}^n e^{(\alpha_1 x_2^j - p_1^j)/\beta_1} + e^{v_1^0/\beta_1}} \\ x_2^0 = \frac{e^{v_2^0/\beta_2}}{\sum_{j=1}^n e^{(\alpha_2 x_1^j - p_2^j)/\beta_2} + e^{v_2^0/\beta_2}} \end{cases}$$

with respect to \mathbf{p}^k . Evaluate at the symmetric price \mathbf{p} with symmetric allocation \mathbf{x} and use the fact that $\frac{\partial(x^1 + \dots + x^n)}{\partial \mathbf{p}^k} = -\frac{\partial \mathbf{x}^0}{\partial \mathbf{p}^k}$, we achieve the following:

$$\frac{\partial \mathbf{x}^0}{\partial \mathbf{p}^k} = \begin{bmatrix} \frac{x_1(1-nx_1)}{\beta_1} & 0 \\ 0 & \frac{x_2(1-nx_2)}{\beta_2} \end{bmatrix} + \begin{bmatrix} \frac{x_1(1-nx_1)}{\beta_1} & 0 \\ 0 & \frac{x_2(1-nx_2)}{\beta_2} \end{bmatrix} \begin{bmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{bmatrix} \frac{\partial \mathbf{x}^0}{\partial \mathbf{p}^k}.$$

(Recall at the symmetric allocation, $x_i^0 = 1 - nx_i, i = 1, 2$). Solving for $\frac{\partial \mathbf{x}^0}{\partial \mathbf{p}^k}$ yields \mathbf{W} as stated in Lemma 2. \square

Proof of Theorem 3: Clearly $(\mathbf{p}^*, \mathbf{x}^*)$ must satisfy the demand equation (20). Moreover Platform 1 choose price vector \mathbf{p}^1 to maximize total profit: $\langle \mathbf{p}^1 - \mathbf{c}, \mathbf{x}^1 \rangle$. The FOCs with respect to \mathbf{p}^1 are

$$\mathbf{x}^1 + \left\{ \frac{\partial \mathbf{x}^1}{\partial \mathbf{p}^1} \right\}' (\mathbf{p}^1 - \mathbf{c}) = \mathbf{0}.$$

Evaluating at the symmetric equilibrium $(\mathbf{p}^*, \mathbf{x}^*)$ and using Lemma 2 yields the following equation:

$$\mathbf{p}^* - \mathbf{c} = \left[\frac{n-1}{n} \mathbf{Y} + \frac{1}{n} \mathbf{W} \right]^{-1} \mathbf{x}^*$$

which, combined with equation (20), fully pin down the equilibrium objects: price \mathbf{p}^* and demand \mathbf{x}^* . \square

Proof of Corollary 3: From equation (20), clearly (1) - (3) are equivalent. The separation property follows directly from Theorem 3. \square

Proof of Corollary 4: Without outside options, the results directly follow from Proposition 5.

With outside options, first we note that

$$\mathbf{W} = \begin{bmatrix} \frac{\beta_1}{x_1^*(1-nx_1^*)} & -\alpha_1 \\ -\alpha_2 & \frac{\beta_2}{x_2^*(1-nx_2^*)} \end{bmatrix}^{-1} = \begin{bmatrix} \beta_1 & -\alpha_1 x_1^*(1-nx_1^*) \\ -\alpha_2 x_2^*(1-nx_2^*) & \beta_2 \end{bmatrix}^{-1} \begin{bmatrix} x_1^*(1-nx_1^*) & 0 \\ 0 & x_2^*(1-nx_2^*) \end{bmatrix}.$$

Since for any finite n , $x_i^* \leq \frac{1}{n}$, so $x_i^*(1-nx_i^*) \rightarrow 0$ as $n \rightarrow \infty$ for $i = 1, 2$. It follows that $\lim_{n \rightarrow \infty} \mathbf{W} = \mathbf{0}$. Therefore, by Theorem 3, we have

$$\lim_{n \rightarrow \infty} \mathbf{p}^* - \mathbf{c} = \lim_{n \rightarrow \infty} \mathbf{Y}^{-1'} \mathbf{x}^* = \lim_{n \rightarrow \infty} \begin{bmatrix} \beta_1 - \alpha_2 x_2^* \\ \beta_2 - \alpha_1 x_1^* \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

\square

B Uniqueness of participation equilibrium

In this section, we present the detailed proof for one of the uniqueness statements mentioned in section 3. The proof utilizes contraction mapping theorem, which can be modified and extended to show other uniqueness results specified in the main paper.

Statement 1. For $n = 2$ (duopoly) and linear externalities, suppose

$$2\{\max_{\theta_i} h_i(\theta_i; 2)\} \left\{ \sum_{j \in \mathcal{S}} |\gamma_{ij}| \right\} < 1, \forall i \in \mathcal{S},$$

for any price profile \mathbf{P} , there exists a unique participation equilibrium.

Proof In this case, we first reformulate the PE condition (3) as

$$\begin{aligned} x_i^1(\mathbf{P}) &= \Pr(\{v_i + \phi_i(\mathbf{x}^1(\mathbf{P})) - p_i^1 + \epsilon_i^1\} \geq \{v_i + \phi_i(\mathbf{x}^2(\mathbf{P})) - p_i^2 + \epsilon_i^2\}) \\ &= 1 - H_i(p_i^1 - p_i^2 - \phi_i(\mathbf{x}^1(\mathbf{P})) + \phi_i(\mathbf{1}_s - \mathbf{x}^1(\mathbf{P})); 2) \end{aligned}$$

for any $i \in \mathcal{S}$, as $\mathbf{x}^2(\mathbf{P}) + \mathbf{x}^1(\mathbf{P}) = \mathbf{1}_s$. To show the uniqueness of PE, it is equivalent to show that for any fixed \mathbf{P} , the system

$$\mathbf{z} = \mathbf{g}(\mathbf{z}) = (g_1(\mathbf{z}), \dots, g_s(\mathbf{z}))$$

has a unique solution in $\mathbf{z} \in [0, 1]^s$, where

$$g_i(\mathbf{z}) = 1 - H_i(p_i^1 - p_i^2 - \phi_i(\mathbf{z}) + \phi_i(\mathbf{1}_s - \mathbf{z}); 2), i \in \mathcal{S}.$$

Note that the Jacobian matrix of the mapping \mathbf{g} is

$$\frac{\partial \mathbf{g}}{\partial \mathbf{z}} = \left(\frac{\partial g_i}{\partial z_j} \right)_{1 \leq i, j \leq s} = \text{Diag}(h_1(\theta_1; 2), \dots, h_s(\theta_s, 2)) 2\Gamma$$

with $\theta_i = p_i^1 - p_i^2 - \phi_i(\mathbf{z}) + \phi_i(\mathbf{1}_s - \mathbf{z}), i \in \mathcal{S}$. The ∞ -norm of this Jacobian matrix $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ has a uniform upper bound $2 \max_{i \in \mathcal{S}} (\{\max_{\theta_i} h_i(\theta_i; 2)\} \{\sum_{j \in \mathcal{S}} |\gamma_{ij}|\})$ for every $\mathbf{z} \in [0, 1]^s$.²⁷ Since this uniform upper bound is strictly less than 1 under the assumption, the mapping $\mathbf{g} = (g_1(\mathbf{z}), \dots, g_s(\mathbf{z})) : [0, 1]^s \rightarrow [0, 1]^s$ is a contraction mapping under ∞ -norm, thus has a unique fixed point. It follows that $\mathbf{z} = \mathbf{g}(\mathbf{z})$ has a unique solution in $[0, 1]^s$. \square

C Proof of the claim that Assumption 2 implies Assumption 1

Suppose that conditions (i) and (ii) in Assumption 2 hold. We want to prove Assumption 1 holds, i.e., to show the concavity of R in \mathbf{z} . To this end, it suffices to check the negative semi-definiteness of the Hessian matrix $(\frac{\partial^2 R}{\partial z_i \partial z_j})_{1 \leq i, j \leq s}$. Notice that

$$\frac{\partial^2 R}{\partial z_i \partial z_j} = Id_{\{i=j\}} \frac{\partial \left\{ z_i H_i^{-1}(1 - z_i; n) \right\}}{\partial z_i^2} + \left\{ 1 + \frac{1}{n-1} \right\} (\gamma_{ij} + \gamma_{ji}). \quad (28)$$

Here $Id_{\{i=j\}}$ is the indicator function, which equals 1 if $i = j$, and 0 otherwise.

For each $i \in \mathcal{S}$, let $D_i(p_i) := 1 - H_i(p_i; n)$. Direct computation shows that

$$\frac{\partial \left\{ z_i H_i^{-1}(1 - z_i; n) \right\}}{\partial z_i^2} = \frac{\partial \left\{ z_i D_i^{-1}(z_i) \right\}}{\partial z_i^2} = -\frac{1}{h_i(p_i; n)} \left(2 - \frac{D_i(p_i) D_i''(p_i)}{[D_i'(p_i)]^2} \right)$$

²⁷For square matrix A , its ∞ -norm is simply the maximum absolute row sum of the matrix, i.e., $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$. The ∞ -norm of a column vector \mathbf{z} is $\|\mathbf{z}\|_\infty = \max_i |z_i|$.

where $p_i = D_i^{-1}(z_i) = H_i^{-1}(1 - z_i; n)$. Since D_i is log-concave, $\frac{D_i(p_i)D_i''(p_i)}{[D_i'(p_i)]^2} \leq 1$, therefore we have ²⁸

$$\frac{\partial \left\{ z_i H_i^{-1}(1 - z_i; n) \right\}}{\partial z_i^2} \leq -\frac{1}{h_i(p_i; n)}. \quad (29)$$

For simplicity, define $\xi_i := \frac{\partial \{z_i H_i^{-1}(1 - z_i; n)\}}{\partial z_i^2} + w_i$, where $\frac{1}{w_i} = \max_{\theta_i} h_i(\theta_i; n)$. Then, by (29),

$$\xi_i \leq \frac{-1}{h_i(p_i; n)} + w_i = \frac{w_i}{h_i(p_i; n)} \overbrace{\left(h_i(p_i; n) - \frac{1}{w_i} \right)}^{\leq 0} \leq 0.$$

Rewrite the Hessian matrix of R as follows

$$\left(\frac{\partial^2 R}{\partial z_i \partial z_j} \right)_{1 \leq i, j \leq s} = \text{Diag}(\xi_1, \dots, \xi_s) - \left\{ \text{Diag}(w_1, \dots, w_s) - \frac{n}{n-1}(\Gamma + \Gamma') \right\}.$$

The diagonal matrix $\text{Diag}(\xi_1, \dots, \xi_s)$ is negative semidefinite as each $\xi_i \leq 0, \forall i \in \mathcal{S}$. Moreover, the matrix $\left\{ \text{Diag}(w_1, \dots, w_s) - \frac{n}{n-1}(\Gamma + \Gamma') \right\}$ is positive definite by item (ii) of Assumption 2. Therefore, the Hessian matrix of R is negative definite at any \mathbf{z} , hence R is concave in $\mathbf{z} \in [0, 1]^s$. \square

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²⁸In particular, it implies that the revenue function $z_i H_i^{-1}(1 - z_i; n)$ is concave in quantity z_i whenever the demand $1 - H_i(p_i)$ is log-concave in price p_i .

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