Complementary Multi-Sided Platforms

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Abstract

Motivated by several examples, including Internet of Things (IoT) patent licensing, we consider a model in which $m \geq 1$ complementary platforms choose prices for $n > 1$ connected devices which generate demand externalities among themselves. We characterize equilibrium prices and show that platforms face a trade-off between an externality internalization effect and a value extraction effect. Given a device, the externality internalization effect (the value extraction effect) represents a weighted sum of all the externalities that a device generates to (receives from) other devices. The weight assigned to each device reflects its position in the network of demand externalities and is measured by the Katz-Bonacich centrality. We show how the centrality measures and resulting pricing decisions change with the number of platforms. Even if Cournot's insight continues to hold (complementary monopolists charge higher prices than an integrated monopolist), surprisingly, the total prices for some particular devices in a duopoly can be lower than the prices in the single monopoly benchmark. We contribute to the two-sided market literature by analyzing complementary platforms in a general multi-sided market.

Key Words: Multi-sided Market, Complementary Platforms, Network, Centrality, IoT, Licensing

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1 Introduction

Motivated by the growth and proliferation of digital intermediaries, a growing body of economic theory analyzes pricing by multi-sided platforms. This literature builds upon a series of seminal papers (Caillaud and Jullien, 2001, 2003; Rochet and Tirole, 2003, 2006; Armstrong, 2006) that, for reasons of tractability and exposition, analyze two-sided platforms. In practice, the leading platforms serve a multitude of sides, to the point where many observers describe them as ecosystems. The prior literature has also focused on two types of pricing: monopoly and competition. With the proliferation of platform business models, however, it is natural that some intermediaries find themselves in complementary rather than competitive relationships.

This paper analyzes a model of complementary ecosystems. We assume linear demand for all goods, but allow for an arbitrary number of platforms and sides, as well as a very general specification of the demand-side externalities among all devices.[1] The model yields answers to a number of novel questions, including: How does a product's position within its ecosystem (network) influence pricing and demand? What are the equilibrium prices charged by complementary platforms that serve partially or completely overlapping user groups? and, How does the presence of a complementary intermediary influence decisions to either subsidize or extract value from a particular side of the platform?

For a monopoly platform, the price of each device reflects the well-known trade-off between internalizing externalities (subsidizing devices that generate larger positive externalities) and extracting value. In our model, these forces are captured by a weighted average of all externalities to/from all other goods, where the weight of each device corresponds to its Katz-Bonacich centrality in the overall demand system. Adding complementary platforms changes the matrix used to compute centrality, such that each platform places more weight on externality internalization relative to value extraction. Using examples, we show how platform complementarity expands the range of equilibrium outcomes relative to the single good case first studied by Cournot (1838). In particular, the total price charged to a single side of the platform can be less than integrated-monopoly benchmark, or exceed the "Cournot price" changed by a pair of complementary monopolists in the absence of demand externalities.

To motivate our model, we use the example of patent licensing for the Internet of Things (IoT). Patent holders have traditionally licensed two sides of the cellular network:

[1] Hereafter, we use the terms good, side, and device interchangeably.

handsets and base stations. To the extent that handset users value greater coverage (i.e. more base stations) and carrier investments reflect the size of the user base, licensors face a two-sided pricing problem. The emergence of IoT, where connected products include not just phones and networks, but also cars, watches, appliances, eyeglasses, and many other goods, converts this into a many side-pricing problem. For a monopolist whose patent portfolio covers all devices, our model yields a particularly simple characterization of optimal pricing. Moreover, our framework can be used to analyze the more realistic scenario of multiple patent holders, whose patents are infringed by some products, but perhaps not others.

This paper contributes to three strands of literature. First, several studies have analyzed two-sided platform competition (Caillaud and Jullien, 2003; Rochet and Tirole, 2003, 2006) and multi-sided monopoly pricing (Weyl, 2010), including the case where users can multi-home (Armstrong and Wright, 2007; Armstrong, 2006; Hagiu, 2009) Within this literature, Van Cayseele and Reynaerts (2011) develop a two-sided model of complementary platforms, where one side must single-home. We analyze a more general model with many (strictly) complementary platforms and inter-group network externalities among $n \geq 2$ sides.

Second, the paper is closely related to a literature on pricing in a network. Fainmesser and Galeotti (2016), Fainmesser and Galeotti (2020) are based on Galeotti et al. (2010)'s random network assumption; firms offer consumers price subsidies for their influence and charge price premia for their susceptibility. Chen et al. (2018), Zhang and Chen (2020),Chen et al. (2020), Bloch and Quérou (2013), Candogan et al. (2012), De Martí and Zenou (2015),Chen et al. (2020) are based on Ballester et al. (2006)'s deterministic model. Bloch and Quérou (2013) and Candogan et al. (2012) prove that the monopoly firm can only price discriminate the consumers when the network is asymmetric. De Martí and Zenou (2015) extend Bloch and Quérou (2013) and Candogan et al. (2012) to incomplete information of agents idiosyncratic characters. Chen et al. (2018) develops a duopoly market on a symmetric network where consumer's utility is affected both by network externalities and by product complementary and firms' pricing are determined by consumers network externalities and the level of complementary. Zhang and Chen (2020) and Chen et al. (2020) are two extensions of Chen et al. (2018): Zhang and Chen (2020) studies asymmetric networks and Chen et al. (2020) studies an oligopoly market. While this literature considers network externalities among consumers, our paper focuses on network externalities among devices (or products).

Finally, we contribute to the literature on patent licensing literature bringing a platform approach to IP licensing that is particularly relevant for the licensing of wireless technology to IoT devices. Katz and Shapiro (1985), Lerner and Tirole (2004), Farrell and Shapiro (2008) and Farrell and Gallini (1988) are seminal papers analysing variant licensing mechanism. Amir et al. (2014) and Choi and Gerlach (2015) are more recent papers focusing on litigation issues.

The remainder of the paper is organized as follows. Section 2 describes the model, provides a micro-foundation for the demand system, and presents the main assumptions. Section 3 analyzes the pricing of a monopoly platform. Section 4 analyzes the equilibrium prices when there are several complementary platforms. Section 5 analyzes welfare-maximizing prices.

2 A Model of Platform Pricing

This section introduces our linear demand system for an ecosystem of many goods linked by positive demand externalities and solves for the prices charged by a monopoly platform sponsor. The analysis yields three broad lessons:

1. Prices reflect a famliiar tradeoff between internalizing externalities and extracting value. When making this tradeoff, the monopolist weights each device by its centrality in a network defined by the demand system.

2. Although the structure of "externality network" matters, the core intuition from two-sided platforms is robust: the monopolist will subsidize products that generate more positive externalities for other goods.

3. When demand externalities are symmetric, they have no impact on pricing, even if they increase overall demand.

2.1 Setup

There are $n > 1$ devices (indexed by i) and $m \geq 1$ platforms (indexed by k) that license to all devices. Let p_i^k denote the royalty charged by platform k to device i. For simplicity, we assume perfectly competitive downstream markets and normalize marginal costs to zero. As a result, the price of each device, i, equals the sum of the royalties charged by the m platforms: $p_i = \sum_{k=1}^{m} p_i^k$.

Connectivity among devices creates externalities in demand. Specifically, we assume that demand for device i is given by

$$D_i = a_i - \sum_{k=1}^{m} p_i^k + \sum_{j \neq i} \alpha_{ij} D_j. \tag{1}$$

where $\alpha_{ij} \geq 0$ captures the strength of the externality exerted by device j's users on the users of device i.[2] For example, the developers of 5G cellular technology will license producers of network infrastructure, smartphones, automobiles, electric meters, and a host of other products (including the software to run them). Increased consumption of any one device in this ecosystem can raise demand for other devices by improving the quality of service or growing the addressable market of complements.

Using matrices, (1) can be written as

$$\mathbf{D} = \mathbf{a} - \mathbf{P1} + \mathbf{AD} \tag{2}$$

where $\mathbf{P}$ is an $n \times m$ matrix and $\mathbf{1}$ is a vector of m elements and

$$\mathbf{A} = \begin{bmatrix} 0 & \alpha_{12} & ... & \alpha_{1n} \\ \alpha_{21} & 0 & ... & \alpha_{2n} \\ ... & ... & ... & ... \\ \alpha_{n1} & \alpha_{n2} & ... & 0 \end{bmatrix} \geq 0$$

Hence, if $\mathbf{I} - \mathbf{A}$ is invertible, the demand system can be written as:

$$\mathbf{D} = (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{a} - \mathbf{P1})$$

If $\lambda_{max}(\mathbf{A})$ represents the largest eigenvalue of $\mathbf{A}$, then a sufficient condition for existence and non-negativity of the matrix $(\mathbf{I} - \mathbf{A})^{-1}$ is that $\lambda_{max}(\mathbf{A}) < 1$.[3] From an economic standpoint, this assumption implies that there are no i, j such that $\partial D_i / \partial p_j^k > 0$, which prevents platform k from setting $p_j^k = \infty$ to achieve infinite profit.[4]

[2] Appendix A provides a micro-foundation for this demand system.

[3] See Theorem III^* of Debreu and Herstein (1953)

[4] Alternatively, we can decompose the demand vector as $\mathbf{D} = \mathbf{I}(\mathbf{a} - \mathbf{p}) + \mathbf{A}(\mathbf{a} - \mathbf{p}) + \mathbf{A^2}(\mathbf{a} - \mathbf{p}) + \cdots$, where $\mathbf{A}^L(\mathbf{a} - \mathbf{p})$ is an L-operator of vector $\mathbf{a} - \mathbf{p}$: a scale transform with less than $\lambda_{max}^L (\geq 0)$ and a rotation transform towards ν_{max}, where $\nu_{max} \geq 0$ is the eigenvector of λ_{max}. Thus, $\mathbf{A}^L(\mathbf{a} - \mathbf{p})$ converges to 0 as L grows larger if and only if $\lambda_{max} < 1$, guaranteeing that the demand system won't explode.

2.2 Monopoly pricing

We begin with the familiar two-sided setting to introduce key concepts, and then characterize monopoly pricing for an ecosystem with n devices.

2.2.1 Two devices

If x represents demand for device 1 and y demand for device 2, then the two-sided demand system is given by

$$\begin{aligned} x &= a_1 - p_1 + \alpha_{12} y \\ y &= a_2 - p_2 + \alpha_{21} x \end{aligned}$$

and solving this system gives

$$\begin{aligned} x &= \frac{a_1 - p_1 + \alpha_{12}(a_2 - p_2)}{1 - \alpha_{12}\alpha_{21}} \\ y &= \frac{a_2 - p_2 + \alpha_{21}(a_1 - p_1)}{1 - \alpha_{12}\alpha_{21}} \end{aligned}$$

Positive externalities create a multiplier effect, given by $(1 - \alpha_{12}\alpha_{21})^{-1}$, through positive feedback. Demand stability therefore requires $\alpha_{12}\alpha_{21} < 1$.

A platform monopolist's total profit is $\pi^M = p_1 x + p_2 y$, and their first-order condition with respect to p_1 is given by

$$x + p_1 \frac{\partial x}{\partial p_1} + p_2 \frac{\partial y}{\partial p_1} = 0$$

or equivalently (after cancelling out the multiplier effect)

$$a_1 + \alpha_{12}(a_2 - p_2) - \alpha_{21} p_2 = 2p_1 \tag{3}$$

To provide some intuition for the monopolist's incentives, we can decompose (3) into three parts:

1. Baseline prices: In the absence of demand externalities (i.e., $\alpha_{21} = \alpha_{12} = 0$), the standard monopoly price is given by $p_1 = a_1/2$. We call this the baseline price(s).

2. Externality internalization: The multi-product monopolist internalizes the effect of

raising p_1 on demand for device 2. This is captured by the term $\frac{\partial y}{\partial p_1} \propto \alpha_{21} > 0$. Externality internalization leads to lower p_1 through the marginal effect on y.

3. Value Extraction: The positive externality from device 2 to device 1 implies that the value of device 1 is enhanced. Specifically, the constant in the demand for device 1 is boosted by $\alpha_{12}(a_2 - p_2) > 0$. This leads the platform to raise p_1. Value extraction occurs not through the marginal effect of changing p_1, but through a level effect.

The two-sided model highlights a tension between *externality internalization* and *value extraction*, which create opposing incentives to reduce or increase the price of each device. We can now ask how these forces play out in a more general setting.

2.2.2 Many devices

In matrix form, the monopolist's system of first-order conditions, each analogous to (3), can be written as

$$\mathbf{a} + \mathbf{A}\left(\mathbf{a} - \mathbf{p}\right) - \mathbf{A}'\mathbf{p} = 2\mathbf{p}. \tag{4}$$

In this expression, $\mathbf{A}\left(\mathbf{a} - \mathbf{p}\right)$ captures the value extraction effect, whereas $\mathbf{A}'\mathbf{p}$ corresponds to externality internalization. Henceforth, we refer to $\mathbf{A}$ as the *value extraction matrix* and $\mathbf{A}'$ as the *externality internalization matrix*.

Appendix B shows that the solution to (4), if one exists, is given by:

$$\mathbf{p}^M = \frac{1}{2}\mathbf{a} + \frac{1}{4}\left(\mathbf{A} - \mathbf{A}'\right)\left[\mathbf{I} - \left(\frac{\mathbf{A} + \mathbf{A}'}{2}\right)\right]^{-1}\mathbf{a}. \tag{5}$$

The first term in (5) is the vector of baseline prices and the second term is a vector of deviations from baseline. The deviations reflect both value extraction and externality internalization, as in the two sided case. Moreover, the extraction and internalization matrices are both post-multiplied by a set of device-specific weights that is well known in the literature on networks. Specifically, we take from that literature

Definition 1 *The $n \times 1$ vector $\left[\mathbf{I} - \left(\mathbf{A} + \mathbf{A}'\right)/2\right]^{-1}\mathbf{a} \equiv \mathbf{c}^{KB}$ measures each device's Katz-Bonacich (KB) centrality in the network $\left(\mathbf{A} + \mathbf{A}'\right)/2$.*

Katz-Bonacich centrality is a commonly used measure of the influence exerted by a particular node in a network. To provide some intuition, let $\mathbf{B} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}'$. KB-centrality can then be decomposed as $\mathbf{c}^{KB,1} = \mathbf{a} + \mathbf{B}\mathbf{a} + \sum_{t=2}^{\infty}\mathbf{B}^{\mathbf{t}}\mathbf{a}$. The term $\mathbf{B}\mathbf{a}$ measures direct centrality: the value of all 1-step links to each device, weighted by $\mathbf{a}$. The term $\sum_{t=2}^{\infty} \mathbf{B^t a}$ measures indirect centrality. It is the sum of the value of all t-step links to a device, where $t = 2, 3, 4, \cdots$, again weighted by $\mathbf{a}$. Indirect centrality is a geometric sequence that will converge if $\lambda_{max}(\frac{\mathbf{A}+\mathbf{A}'}{2}) < 1$. The same condition guarantees that demand is well-behaved.[5] Thus, we have

Theorem 1 *If $\lambda_{max}(\frac{\mathbf{A}+\mathbf{A}'}{2}) < 1$, then there exists a unique vector of optimal monopoly prices given by*

$$\mathbf{p}^M = \frac{1}{2}\left[\mathbf{a}+\frac{1}{2}\left(\mathbf{A}-\mathbf{A}'\right)\mathbf{c}^{KB}\right]. \tag{6}$$

Equation (6) shows how demand externalities create a trade-off between value extraction and externality internalization for the monopolist. We refer to the two matrices $\mathbf{A}\mathbf{c}^{KB}$, and $\mathbf{A}'\mathbf{c}^{KB}$ as in-degree and out-degree KB-centrality respectively. As a device's in-degree centrality increases, it has a higher demand so the monopoly chooses a higher price. As a device's out-degree centrality increases, it exerts stronger externalities to other devices and the monopoly chooses a lower price. When $\mathbf{A}$ is symmetric these two incentives are in perfect balance, leading to

Corollary 1 *For symmetric demand externalities, $\mathbf{A} = \mathbf{A}'$, a monopolist charges the baseline prices $\mathbf{p}^M = \frac{1}{2}\mathbf{a}$.*

To solve for demand under monopoly pricing, we can substitute the prices from (6) into the demand system (2), which yields

$$(\mathbf{I}-\mathbf{A})\mathbf{D} = \mathbf{a} - \mathbf{p}^M = \frac{1}{2}\mathbf{a} - \frac{1}{4}\left(\mathbf{A}-\mathbf{A}'\right)\mathbf{c}^{KB}.$$

Adding $\frac{1}{2}\mathbf{A}\mathbf{c}^{KB}$ to both sides of the equation and using the definition of $\mathbf{c}^{KB}$, this equality simplifies to

$$\begin{aligned}(\mathbf{I}-\mathbf{A})\mathbf{D} + \frac{1}{2}\mathbf{A}\mathbf{c}^{KB} &= \frac{1}{2}\mathbf{a} + \frac{1}{4}\left(\mathbf{A}+\mathbf{A}'\right)\mathbf{c}^{KB} = \frac{1}{2}\mathbf{c}^{KB} \\ (\mathbf{I}-\mathbf{A})\mathbf{D} &= (\mathbf{I}-\mathbf{A})\mathbf{c}^{KB}/2 \\ \mathbf{D} &= \mathbf{c}^{KB}/2,\end{aligned}$$

and we restate this result as

[5] See theorem 10.28 of Zhang(2011).

Theorem 2 *For linear demand with monopoly pricing, quantities are proportional to the KB-centrality of each device, with constant of proportionality* $\frac{1}{2}$.

Theorem 2 says that a monopolist sells more of a device when that device is more central in the network of demand externalities. This finding may rationalize, for example, efforts by large platforms (e.g., Amazon, Apple, Google) to ensure broad adoption of the core devices within their "Smart Home" ecosystems (i.e., Echo/Alexa, HomePod/Siri, and Nest/Assistant respectively).

Thus far, our analysis shows that the relevant notion of centrality for a monopolist is based on a weighted average of in- and out-degree KB-centrality. Below, we show the weights change for a social planner or in the presence of complementary platforms. Before considering those settings, however, we pause to link our characterization of monopoly pricing to prior literature and solve some examples.

2.3 Comparison with Armstrong (2006)

While demand in Armstrong (2006) is equivalent to (1), he uses a change of variable to express this in terms of utility for each device,

$$u_i = \sum_{j \neq i} \alpha_{ij} D_j - p_i, \tag{7}$$

so quantities are given by $D_i = a_i + u_i$. The platform's profit is $\Pi = \sum p_i D_i$, and its first-order condition with respect to u_i (holding D_j for all $j \neq i$ constant) is therefore

$$\sum_{j \neq i} \alpha_{ij} D_j - u_i - D_i + \sum_{j \neq i} \alpha_{ji} D_j = 0. \tag{8}$$

Rearranging the first-order condition gives the generalized Armstrong pricing rule

$$\frac{p_i + \sum_{j \neq i} \alpha_{ji} D_j}{p_i} = \frac{1}{\varepsilon_i}$$

where $\varepsilon_i = -\frac{\partial D_i}{\partial p_i} / \frac{D_i}{p_i} = p_i / D_i$. The appearance of demand externalities where we would normally observe marginal costs in the Lerner (1934) markup rule highlights the additional (opportunity) cost of lost revenue from sales of other devices.

By substituting (7) into (8) we can derive a *modified* Armstrong pricing formula

$$p_i = \frac{a_i}{2} + \frac{1}{2} \sum_{j \neq i} (\alpha_{ij} - \alpha_{ji}) D_j$$

that shows how p_i can be decomposed into (i) the baseline price, (ii) a value extraction term, and (iii) an externality internalization term. This formula, however, expresses p_i as a function of D_j, an endogenous variable. Our own characterization of the monopoly pricing takes the same shape but expresses D_j in terms of the fundamentals:

$$p_i = \frac{a_i}{2} + \frac{1}{4} \sum_{j \neq i} (\alpha_{ij} - \alpha_{ji}) c_j^{KB}$$

Setting these two expressions for p_i equal to one another reveals, again, that $D_i = c_i^{KB}/2$.

2.4 Examples

To illustrate how a monopolist would price different device "ecosystems" we consider three simple examples. In each example, the externality between any pair of devices takes one of three values, $\alpha_{ij} \in \{\mu, \eta, 0\}$. We set all of the demand intercepts $a_i = 1$, and define two parameters $c \equiv \mu + \eta$ and $d \equiv \mu - \eta$. Finally, we will say that a device is subsidized (respectively, exploited) if its price is lower (respectively, higher) than its baseline price.

2.4.1 Star

A star network is defined by $\alpha_{1j} = \eta$ for all $j > 1$; $\alpha_{j1} = \mu$ for all $j > 1$; and $\alpha_{ij} = 0$ for all $i, j > 1$. For this demand system all of the externalities either originate from or terminate at the "star" device ($i = 1$). In terms of our licensing example, one might think of the star as a smartphone that exhibits bilateral demand externalities with a series of other applications, such as watches, cars, thermostats, eyeglasses, etc. that do not interact with one another.

Using (6) and fact that all of the peripheral devices ($j > 1$) are symmetric, we can write the monopoly prices as

$$p_1^M = \frac{1}{2} - \frac{1}{4} d(n-1) c_j^{KB}$$
$$p_j^M = \frac{1}{2} + \frac{1}{4} d c_1^{KB}$$

Because the KB-centrality of each device, c^{KB}, is strictly positive, we see that the star device will be subsidized if and only if $d > 0$ (i.e., when its *net* externalities to each peripheral are positive). These price formula also reveal that when the star device is subsidized, the peripherals are exploited, and vice versa.

In Appendix C, we analyze a more general version of the star network, where strength of the demand externalities can be different for each peripheral. In that model, we can derive a similar result: the star device is subsidized if and only if the aggregate externalities that it creates for peripherals exceed the aggregate externalities generated by all peripheral devices to the star.

2.4.2 Hierarchy

Next, consider a "hierarchical" ecosystem of devices, where $\alpha_{ij} = \eta$ for all $i < j$, and $\alpha_{ij} = \mu$ for all $i > j$. When $\mu > \eta$, the device 1 generates the most and receives the fewest externalities, device 2 generates the second-most and receives the second-least amount of externalities, and so on. In economic terms, this example corresponds to a setting there are some devices that clearly produce more externalities than others, but there is no single dominant device or side to the platform.

For this demand system, every non-diagonal element in the matrix $[\mathbf{I} - (\mathbf{A} + \mathbf{A}')/2]$ equals $\frac{c}{2}$, and because its inverse exhibits the same symmetry, all devices have the same KB-centrality. This fact, together with (6), yields monopoly prices for each device

$$p_i^M = \frac{1}{2} - \frac{d}{4}\,(n + 1 - 2i)\, c^{KB}.$$

This expression implies that when $d > 0$, a monopolist will subsidize devices that are "higher" in the hierarchy ($i < \frac{n+1}{2}$) and exploit the devices that are "lower" in the hierarchy. For devices near the middle of the hierarchy, which generate and receive similar amounts of externalities, prices will be close to the monopoly baseline. Subsidies or surcharges are greatest for devices near the top ($i = 1$) or bottom ($i = n$). It is also worth emphasizing that in this example, all of the price distortions reflect the trade-off between externality internalization and value extraction, as captured by $[\mathbf{A} - \mathbf{A}']$, given that every device has the same KB-centrality.

2.4.3 Ring

As a final example, we consider a demand system with "circular" externalities represented by $\alpha_{ij} = \mu$ if $i = j - 1$ (or $i = n$ and $j = 1$); $\alpha_{ij} = \eta$ if $i = j + 1$ (or $i = 1$ and $j = n$); and otherwise $\alpha_{ij} = 0$. In this example, each device has two neighbors, one of which receives μ and creates η, while the other receives η and creates μ for the focal device. Although we are not aware of any actual ecosystems that exhibit this type of circularity, the example remains useful for developing intuition.

As in the previous example of a hierarchical demand system, all devices in the ring have the same KB-Centrality. Moreover, each row in $[\mathbf{A} - \mathbf{A}']$ has exactly one entry entry equal to d, one equal to $-d$, and the rest equal to zero. Therefore, applying (6) reveals that

$$p_i^M = \frac{1}{2} + \frac{1}{4}(dc^{KB} - dc^{KB}) = \frac{1}{2}.$$

The monopoly platform sponsor selects baseline prices in this example because, although $\mathbf{A}$ is not symmetric, the circular structure implies that the *net* externalities produced by each device are zero.

3 Complementary Platforms

We now consider a model with m platforms that supply perfectly complementary inputs to each of the n devices. In a licensing context, these inputs could represent a portfolio of IP rights held by m distinct licensors that are essential for the production of all n devices. This is roughly the situation faced by participants in the licensing market for 5G Standard Essential Patents used in various "Internet of Things" applications.[6]

3.1 General Case: n devices and m platforms

Recall that p_i^k is the price charged by platform k to device i, and $p_i = \sum_{k=1}^{m} p_i^k$. Let $\mathbf{p}^k = \left(p_1^k, p_2^k, ..., p_n^k\right)'$ represent the vector of prices charged by platform k and $\mathbf{P} = (p_1, p_2, ..., p_n)'$ represent the vector of total prices (input costs) collectively charged by the m platforms to each device. Maintaining the assumption that the downstream market is perfectly

[6] In actual 5G licensing, most of the licensing "platfroms" have made commitments to license their patents on Fair Reasonable and Non-Discriminatory (FRAND) terms. In our analysis, we simply treat each platform as a monopoly input supplier, thereby ignoring any FRAND pricing constraints.

competitive, platform k's profit is given by

$$\mathbf{\Pi}^k = \mathbf{p}^{k\prime}(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{a}-\mathbf{P}).$$

To solve for the symmetric equilibrium prices charged by all platforms to each device, we differentiate this expression with respect to $\mathbf{p}^k$ and aggregate the system of first-order conditions. These computations, found in Appendix D, show that the vector of prices charged by each of the m platforms is

$$\mathbf{p}^* = \frac{1}{m+1}\left[\mathbf{a}+\frac{1}{(m+1)}(\mathbf{A}-\mathbf{A}')\left[\mathbf{I}-\frac{1}{m+1}\mathbf{A}-\frac{m}{m+1}\mathbf{A}'\right]^{-1}\mathbf{a}\right] \tag{9}$$

Let the parameter $\lambda = 1/(m+1)$, so the first term in (9) equals $\lambda\mathbf{a}$. This is the price charged by each one of m independent monopolists in Cournot's famous complementary monopolies problem. Henceforth, we refer to these as the Cournot baseline.

The second term in (9) contains the matrix $(\mathbf{A}-\mathbf{A}')$. As in the monopoly case, this matrix reflects a tradeoff between value capture (through $\mathbf{A}$) and externality internalization (through $\mathbf{A}'$). The second term differs from the monopoly formulas for subsidy/exploitation, however, in the device-specific weights that post-multiply $(\mathbf{A}-\mathbf{A}')$. We therefore introduce

Definition 2 *The $n\times 1$ vector $[\mathbf{I}-(\mathbf{A}+m\mathbf{A}')/(m+1)]^{-1}\mathbf{a}\equiv\mathbf{c}^{KB,m}$ measures each device's Katz-Bonacich centrality in the network $(\mathbf{A}+m\mathbf{A}')/(m+1)$.*

We refer to the i^{th} component of $c^{KB,m}$ as device i's KB-m centrality. Compared to the monopoly case, the network used to calculate KB-m centrality places more weight on externality internalization. Intuitively, as we add more monopoly input suppliers, the value-capture incentive declines because each firm's residual demand curve shifts inward (i.e. the demand intercept for any single firm shifts from a_i to $a_i-\sum_{j\neq k}p_i^k$). The internalization incentive, however, remains unchanged because it reflects a marginal effect and not the level of demand. Thus, as m increases, the network used to compute KB-m centrality places increased weight on internalization. We summarize the general expression for symmetric equilibirum pricing in

Theorem 3 *If $\lambda_{max}(\frac{\mathbf{A}+m\mathbf{A}'}{m+1})<1$, then the unique vector of symmetric equilibrium prices* *charged by each of m complementary platforms is given by*

$$\mathbf{p}^* = \lambda \left[\mathbf{a} + \lambda \left(\mathbf{A} - \mathbf{A}'\right) \mathbf{c}^{KB,m}\right]. \tag{10}$$

And for a symmetric demand system, the second term in (10) disappears, so we have

Corollary 2 *If the network externalities are symmetric (i.e., $\mathbf{A} = \mathbf{A}'$), then equilibrium prices are equal to the Cournot baseline $\mathbf{p}^* = \lambda\mathbf{a}$.*

3.2 Cournot Comparisons

With multiple platform sponsors, the baseline prices suffer from double marginalization. That is, aggregate input prices ($m\lambda = 1 - \lambda$) exceed the monopoly level ($\frac{1}{2}$), so the m suppliers would profit from a coordinated price reduction. In the context of patent licensing, double marginalization is often called *royalty stacking*, and it is frequently offered as a justification for joint licensing programs (e.g. through patent pools).

As we have just seen, however, the prices charged by complementary platforms will also reflect incentives to internalize demand externalities. To illustrate how this may alter standard Cournot results we return to the example of a star network, where device 1 (the star) generates an externality μ to each peripheral ($j > 1$), and conversely, the peripherals generate η to the star.

In appendix C, we using (10) to compute the symmetric equilibrium prices, which are

$$\begin{aligned} p_1^* &= \lambda - \lambda^2 \Delta d(n-1)\left(1 + \frac{1}{2}c - \frac{1-2\lambda}{2}d\right) \\ p_k^* &= \lambda + \lambda^2 \Delta d(n-1)\left(\frac{1}{(n-1)} + \frac{1}{2}c + \frac{1-2\lambda}{2}d\right). \end{aligned}$$

where $\Delta^{-1} = 1 - \frac{(n-1)}{4}(c^2 - (1-2\lambda)^2 d^2) > 0$. These prices imply that the star device is subsidized and the peripherals exploited if and only if $d > 0$.

The fundamental Cournot result is that increasing m leads each supplier to charge a lower price, $\lambda = \frac{1}{m+1}$, but still generates a higher total downstream cost $\frac{m}{m+1} = 1 - \lambda$. Thus, we seek examples where adding platforms leads to a decline in the aggregate device price. For the simple star network, it is possible to prove that

Theorem 4 *For a symmetric star network with $d > 0$, served by m strictly complementary platforms*

- *The total amount of subsidy to the star device $m(\lambda - p_1^*)$ decreases with m, which implies that the total price of the star device increases with m*

- *The total amount of exploitation for each peripheral $m(p_k^* - \lambda)$ decreases with m, but mp_k^* increases with m.*

Proof. See appendix E. ■

For the star device, double marginalization raises the baseline price and also reduces the total subsidy. These two effects work together, so the total price of the star device increases with m by more than in the simple Cournot model. For the peripheral device, increasing m raises the baseline price and reduces the amount of exploitation. Although these factors push in opposing directions, the former is larger, so the total price of the peripheral will still increase with m.

Although this example does not contradict the qualitative results of the standard Cournot model, it does suggests a way to do so. Recall that the amount of exploitation for peripherals depends on the centrality of the star device, and vice versa. So, in order to make mp_k^* decline with m we might try increasing the centrality of the star (so that it declines more quickly as m grows).

To implement this idea, consider the same demand system, but let the baseline demand for the star device exceed that of the peripherals. In particular, suppose $a_1 \equiv \beta > 1$, $a_2 = a_3 = \cdots = 1$. For this augmented star network the total price of each peripheral under monopoly is

$$p_k^M = \frac{1}{2} + \frac{1}{4} d c_1^{KB,1} = \frac{1}{2} + \frac{1}{4} d \Delta_1 (\beta + \frac{1}{2} c(n-1))$$

and the total price of a peripheral when $m = 2$ is:

$$2p_k^2 = \frac{2}{3} + \frac{2}{9} d c_1^{KB,2} = \frac{2}{3} + \frac{2}{9} d \Delta_2 (\beta + \frac{c(n-1)}{2} + \frac{1}{6} d(n-1))$$

and comparing these two expressions reveals that

Theorem 5 *For the augmented star network, when $\beta > \frac{5}{3}\sqrt{n-1}$, the total price of a peripheral device is smaller when $m = 2$ than when $m = 1$.*

Proof. See Appendix F ■

This example shows that incentives to internalize demand-side externalities can be strong enough to completely overturn the Cournot double-marginalization problem, at least for a single device viewed in isolation. Intuitively, increasing m leads to a decline in $c_1^{KB,m}$ because each platform puts more weight on externality internalization. For a star network, the degree of exploitation of each peripheral is proportional to $c_1^{KB,m}$, and the marginal impact of a decline in star centrality will increase with β. This example shows that for sufficient β these changes in platfrom pricing can fully offset an increase in the baseline prices.

4 Overlapping Platforms

In the last section, we assumed that each of the m platforms provided a necessary input to all of the n devices. However, one can imagine cases where the customer groups served by different platforms are only partially overlapping. For example, in the licensing context one firm may hold IP rights connecting phones to cars and another may hold IP rights connecting phones to appliances. Another example comes from mobile gaming and the recent antitrust lawsuit between Apple and EPIC. In that setting, EPIC and Apple both collect fees from game-players, and at the same time EPIC serves developers and Apple sells the gaming device.[7] This section considers pricing in similar contexts, where platforms are strict complements for one set of devices (or sides) and monopolists for another.

4.1 A Star Network with $m = 2$ and $n = 3$

Suppose there are $n = 3$ devices, and $m = 2$ platforms. By definition, the platforms are complements for the star device, but each has a monopoly on its other side. Thus, we have

$$\mathbf{A} = \begin{pmatrix} 0 & \eta & \eta \\ \mu & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}; \quad \mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & 0 \\ 0 & p_{32} \end{pmatrix}$$

The demand system is:

$$\mathbf{D} = (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{a} - \mathbf{P1})$$

and stability of demand requires that $c^2 < \frac{1}{2}$.

[7]The Japanese game developer GREE also exhibits partial overlap with device makers, as described in the HBS case by Andrei Hagiu.

Symmetry implies that $p^*_{11} = p^*_{12}$, $p^*_{21} = p^*_{32}$, and those conditions can be used to calculate the equilibrium prices

$$p^*_{11} = \frac{1}{3} + \frac{3c(2-c^2) + d(3d^2 + (2-3c)d - 3(c^2+2c+6))}{12(6-3c^2+d^2)}$$
$$p^*_{21} = \frac{1}{2} + \frac{d(c+1)}{6-3c^2+d^2}$$

For simplicity, we focus on the special case where $\mu = \eta$ (or equivalently, $d = 0$) so the equilibrium prices simplify to

$$p^*_{11} = \frac{1}{3} + \frac{c}{12},\ p^*_{21} = \frac{1}{2}$$

Recall that when $\mathbf{A} = \mathbf{A}'$, as it does here, the fully overlapping platforms charge the baseline prices $\lambda = \frac{1}{3}$ and a monopoly also charges the baseline price $\frac{1}{2}$. In the partially overlapping case, though, each platform charges the monopoly baseline for its monopolized device, and a price that *exceeds* the Cournot baseline on the overlapping device. Put differently, platform overlap exacerbates the double marginalization problem (in this example) for the overlapped device.

Once again, the intuition for this result emerges from the tradeoff between internalizing externalities and extracting value. When there is only partial overlap, neither of the two platforms will fully internalize the benefits that consumption of the shared device generates for the other platform. This leads both platforms to raise prices on the shared device above the fully overlapping baseline price level.

5 Conclusions

TBD

References

Ambrus, A., E. Calvano, and M. Reisinger (2016). Either or both competition: A" two-sided" theory of advertising with overlapping viewerships. *American Economic Journal: Microeconomics 8*(3), 189–222.

Amir, R., D. Encaoua, and Y. Lefouili (2014). Optimal licensing of uncertain patents in the shadow of litigation. *Games and Economic Behavior 88*, 320–338.

Anderson, S. P. and S. Coate (2005). Market provision of broadcasting: A welfare analysis. *The review of Economic studies 72*(4), 947–972.

Armstrong, M. (2006). Competition in two-sided markets. *The RAND journal of economics 37*(3), 668–691.

Armstrong, M. and J. Wright (2007). Two-sided markets, competitive bottlenecks and exclusive contracts. *Economic Theory 32*(2), 353–380.

Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006). Who's who in networks. wanted: The key player. *Econometrica 74*(5), 1403–1417.

Bloch, F. and N. Quérou (2013). Pricing in social networks. *Games and economic behavior 80*, 243–261.

Caillaud, B. and B. Jullien (2001). Competing cybermediaries. *European Economic Review 45*(4-6), 797–808.

Caillaud, B. and B. Jullien (2003). Chicken & egg: Competition among intermediation service providers. *RAND journal of Economics*, 309–328.

Candogan, O., K. Bimpikis, and A. Ozdaglar (2012). Optimal pricing in networks with externalities. *Operations Research 60*(4), 883–905.

Chen, Y.-J., Y. Zenou, and J. Zhou (2018). Competitive pricing strategies in social networks. *The RAND Journal of Economics 49*(3), 672–705.

Chen, Y.-J., Y. Zenou, and J. Zhou (2020). Network topology and market structure.

Choi, J. P. and H. Gerlach (2015). Patent pools, litigation, and innovation. *The RAND Journal of Economics 46*(3), 499–523.

De Martí, J. and Y. Zenou (2015). Network games with incomplete information. *Journal of Mathematical Economics 61*, 221–240.

Debreu, G. and I. N. Herstein (1953). Nonnegative square matrices. *Econometrica: Journal of the Econometric Society*, 597–607.

Fainmesser, I. P. and A. Galeotti (2016). Pricing network effects. *The Review of Economic Studies 83*(1), 165–198.

Fainmesser, I. P. and A. Galeotti (2020). Pricing network effects: Competition. *American Economic Journal: Microeconomics 12*(3), 1–32.

Farrell, J. and N. T. Gallini (1988). Second-sourcing as a commitment: Monopoly incentives to attract competition. *The Quarterly Journal of Economics 103*(4), 673–694.

Farrell, J. and C. Shapiro (2008). How strong are weak patents? *American Economic Review 98*(4), 1347–69.

Galeotti, A., S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv (2010). Network games. *The review of economic studies 77*(1), 218–244.

Galeotti, A. and F. Vega-Redondo (2011). Complex networks and local externalities: A strategic approach. *International Journal of Economic Theory 7*(1), 77–92.

Hagiu, A. (2009). Two-sided platforms: Product variety and pricing structures. *Journal of Economics & Management Strategy 18*(4), 1011–1043.

Katz, M. L. and C. Shapiro (1985). On the licensing of innovations. *The RAND Journal of Economics*, 504–520.

Lerner, J. and J. Tirole (2004). Efficient patent pools. *American Economic Review 94*(3), 691–711.

Rochet, J.-C. and J. Tirole (2003). Platform competition in two-sided markets. *Journal of the european economic association 1*(4), 990–1029.

Rochet, J.-C. and J. Tirole (2006). Two-sided markets: a progress report. *The RAND journal of economics 37*(3), 645–667.

Sundararajan, A. (2006). Local network effects and complex network structure. *Available at SSRN 650501*.

Ushchev, P. and Y. Zenou (2020). Social norms in networks. *Journal of Economic Theory 185*, 104969.

Van Cayseele, P. and J. Reynaerts (2011). Complementary platforms. *Review of Network Economics 10*(1).

Weyl, E. G. (2010). A price theory of multi-sided platforms. *American Economic Review 100*(4), 1642–72.

Zhang, F. (2011). *Matrix Theory: Basic Results and Techniques*. Universitext. Springer New York.

Zhang, Y. and Y.-J. Chen (2020). Optimal nonlinear pricing in social networks under asymmetric network information. *Operations Research 68*(3), 818–833.

Appendices

A Microfoundation for the demand system

Consider a unit-mass of heterogenous consumers indexed by $\theta \in [0,1]$. Denote p_i the price of device i and N_i the mass of consumers buying device i. We assume that the utility of consumer $\theta \in [0,1]$ is given by

$$u^\theta = \sum_i u_i^\theta$$

where

$$u_i^\theta = a_i^\theta - p_i + \sum_{j \neq i} \alpha_{ij} N_j$$

is the utility obtained by the consumer from using device i. The parameter $\alpha_{ij} \geq 0$ captures the network externality exerted by the users of device j on the users of device i.

For the sake of simplicity, we assume that $a_1^\theta, a_2^\theta, ..., a_n^\theta$ are not correlated for any $\theta \in [0,1]$ and that a_i^θ is uniformly distributed over an interval $[\underline{a}_i, \overline{a}_i]$ where $\underline{a}_i < \overline{a}_i$. Assuming that $a_1^\theta, a_2^\theta, ..., a_n^\theta$ are not correlated for any $\theta \in [0,1]$ implies that there are no complementarities between the devices at the individual level. In other words, network externalities are the only source of complementarities.

For given expectations N_j, $j \neq i$, the demand for device i is

$$\begin{aligned} D_i &= \Pr[u_i^\theta \geq 0] \\ &= \Pr[a_i^\theta \geq p_i - \sum_{j \neq i} \alpha_{ij} N_j] \\ &= \frac{\overline{a}_i - p_i + \sum_{j \neq i} \alpha_{ij} N_j}{\overline{a}_i - \underline{a}_i} \end{aligned}$$

over the range of prices for which this expression is between 0 and 1.

It is sufficient to normalize the difference $\overline{a}_i - \underline{a}_i$ to 1 and denote $\overline{a}_i = a_i$ to obtain

$$D_i = a_i - p_i + \sum_{j \neq i} \alpha_{ij} N_j.$$

Therefore, in a fullfilled expectation equilibrium, the demand system for the n devices satisfies

$$D_i = a_i - p_i + \sum_{j \neq i} \alpha_{ij} D_j.$$

Since the price of device i is $p_i = \sum_{k=1}^{m} p_i^k$, we get

$$D_i = a_i - \sum_{k=1}^{m} p_i^k + \sum_{j\neq i} \alpha_{ij} D_j.$$

Importantly, the above microfoundation can be extended to the case in which each consumer may only be interested in a subset of devices. This follows easily from our assumption that $a_1^\theta, a_2^\theta, ..., a_n^\theta$ are not correlated for any $\theta \in [0,1]$.

B Derivation of Monopoly Platform Prices in (5)

Let $\mathbf{B} = \mathbf{I} - \mathbf{A}$. The monopolist's profit is given by

$$\mathbf{\Pi}^M = \mathbf{p}'\mathbf{B}^{-1}(\mathbf{a} - \mathbf{p}).$$

The first-order condition associated with the maximization of $\mathbf{\Pi}^M$ with respect to $\mathbf{p}$ is

$$\mathbf{B}^{-1}(\mathbf{a} - \mathbf{p}) - \left(\mathbf{B}^{-1}\right)'\mathbf{p} = \mathbf{0},$$

which leads to

$$\begin{aligned}
\mathbf{p} &= \left(\mathbf{B}^{-1} + \mathbf{B}'^{-1}\right)^{-1}\mathbf{B}^{-1}\mathbf{A} = \left(\mathbf{B}\mathbf{B}^{-1} + \mathbf{B}\mathbf{B}'^{-1}\right)^{-1}\mathbf{a} \\
&= (\mathbf{I} + \mathbf{B}\mathbf{B}'^{-1})^{-1}\mathbf{A} = \left[2\mathbf{I} + (\mathbf{B} - \mathbf{B}')\mathbf{B}'^{-1}\right]^{-1}\mathbf{a}.
\end{aligned}$$

Using the formula $(\mathbf{X} + \mathbf{Y})^{-1} = \mathbf{X}^{-1} - \mathbf{X}^{-1}\left(\mathbf{X}^{-1} + \mathbf{Y}^{-1}\right)\mathbf{X}^{-1}$ and conducting simple algebraic manipulations yields

$$\begin{aligned}
\mathbf{p} &= \left[\frac{1}{2}\mathbf{I} - \frac{1}{2}\left[\mathbf{I} + 2\mathbf{B}'(\mathbf{B} - \mathbf{B}')^{-1}\right]^{-1}\right]\mathbf{a} \\
&= \left[\frac{1}{2}\mathbf{I} - \frac{1}{2}\left[(\mathbf{B} - \mathbf{B}' + 2\mathbf{B}')(\mathbf{B} - \mathbf{B}')^{-1}\right]^{-1}\right]\mathbf{a} \\
&= \left[\frac{1}{2}\mathbf{I} - \frac{1}{2}(\mathbf{B} - \mathbf{B}')(\mathbf{B} + \mathbf{B}')^{-1}\right]\mathbf{a}
\end{aligned}$$

Using $\mathbf{B} = \mathbf{I} - \mathbf{A}$, we can rewrite the monopoly prices as

$$\mathbf{p}^M = \frac{1}{2}\mathbf{a} + \frac{1}{4}\left(\mathbf{A} - \mathbf{A}'\right)\left[\mathbf{I} - \left(\frac{\mathbf{A} + \mathbf{A}'}{2}\right)\right]^{-1}\mathbf{a}.$$

C Analysis of More General Star Network

A star network is defined by

$$\mathbf{A} = \begin{pmatrix} 0 & \boldsymbol{\eta}' \\ \boldsymbol{\mu} & \mathbf{0} \end{pmatrix},$$

where $\boldsymbol{\mu}' = (\mu_2, ..., \mu_n)$ and $\boldsymbol{\eta}' = (\eta_2, ..., \eta_n)$.

It is useful to define two vectors $\mathbf{c} \equiv \boldsymbol{\mu} + \boldsymbol{\eta}$ and $\mathbf{d} \equiv \boldsymbol{\mu} - \boldsymbol{\eta}$, and to let d_k represent the element in the k^{th} row of $\mathbf{d}$. The elements of $\mathbf{c}$ are (weakly) positive, and correspond to the *total* externalities between a pair of devices, whereas d_k might take either sign and represents the *net* externality from the star to device k. Using these definitions, we can prove that

Proposition 1 *Monopoly prices for a star network are given by:*

$$p_1^M = \frac{1}{2} - \frac{1}{4}\Delta\mathbf{d}'(\mathbf{1} + \frac{1}{2}\mathbf{c} + \frac{1}{4}(\mathbf{c}\mathbf{c}'\mathbf{1} - \mathbf{1}\mathbf{c}'\mathbf{c})); \tag{C.1}$$

$$p_k^M = \frac{1}{2} + \frac{1}{4}\Delta d_k(1 + \frac{1}{2}\mathbf{c}'\mathbf{1}), \text{ for } k = 2, ..., n, \tag{C.2}$$

where the scalar $\Delta = (1 - \mathbf{c}'\mathbf{c}/\mathbf{4})^{-1}$.

Using equations (C.1) and (C.2), and assuming that $c_j = c$ for all $j \geq 2$ for simplicity, we can solve for the star network monopoly prices, which are

$$p_1^M = \frac{1}{2} - \frac{1}{4}\Delta(\Sigma_{k=2}^n d_k)(1 + \frac{1}{2}c)$$

$$p_k^M = \frac{1}{2} + \frac{1}{4}\Delta d_k(1 + \frac{1}{2}c(n-1)))$$

The first equation reveals that the star device is subsidized if and only if $\Sigma_{k=2}^n d_k > 0$. That is, a necessary and sufficient condition for subsidizing the star device in this example is that aggregate externalities generated by the star to all peripheral devices exceed aggregate externalities generated by all peripheral devices to the star. The second equation indicates that a peripheral device k is subsidized if and only if $d_k < 0$, which implies that it creates stronger externalities for the star than vice versa.

D Derivation of Prices in (9)

The first-order condition associated with the maximization of $\mathbf{\Pi}^k$ with respect to $\mathbf{p}^k$ is

$$\mathbf{B}^{-1}(\mathbf{a}-\mathbf{P})-\left(\mathbf{B^{-1}}\right)'\mathbf{p}^k=\mathbf{0}.$$

In a symmetric equilibrium $\mathbf{p}^k=\mathbf{p}^*$ for $k=1,...,m$, we have

$$\mathbf{B}^{-1}(\mathbf{a}-m\mathbf{p}^*)-\left(\mathbf{B^{-1}}\right)'\mathbf{p}^*=\mathbf{0},$$

which leads to

$$\begin{aligned}
\mathbf{p}^* &= \left(m\mathbf{B^{-1}}+\mathbf{B'^{-1}}\right)^{-1}\mathbf{B^{-1}a} \\
&= \left(m\mathbf{I}+\mathbf{BB'^{-1}}\right)^{-1}\mathbf{a} \\
&= \left[m\mathbf{I}+\left(\mathbf{B}-\mathbf{B}'+\mathbf{B}'\right)\mathbf{B'^{-1}}\right]^{-1}\mathbf{a} \\
&= \left[(m+1)\,\mathbf{I}+\left(\mathbf{B}-\mathbf{B}'\right)\mathbf{B'^{-1}}\right]^{-1}\mathbf{a}
\end{aligned}$$

Using again the formula $(\mathbf{X}+\mathbf{Y})^{-1}=\mathbf{X}^{-1}-\mathbf{X}^{-1}\left(\mathbf{X}^{-1}+\mathbf{Y}^{-1}\right)\mathbf{X}^{-1}$ and conducting simple algebraic manipulations yields

$$\mathbf{p}^*=\frac{1}{m+1}\mathbf{a}+\frac{1}{(m+1)^2}\left(\mathbf{B}'-\mathbf{B}\right)\left[\frac{1}{m+1}\mathbf{B}+\frac{m}{m+1}\mathbf{B}'\right]^{-1}\mathbf{a}.$$

Using $\mathbf{B}=\mathbf{I}-\mathbf{A}$, we can rewrite the monopoly prices

$$\mathbf{p}^*=\frac{1}{m+1}\mathbf{a}+\frac{1}{(m+1)^2}\left(\mathbf{A}-\mathbf{A}'\right)\left[\mathbf{I}-\frac{1}{m+1}\mathbf{A}-\frac{m}{m+1}\mathbf{A}'\right]^{-1}\mathbf{a}$$

E Proof for Theorem 4

First we proof $m(\lambda-p_1^*)$ is decreasing with m:

$$m(\lambda-p_1^*)=\lambda(1-\lambda)\Delta d(n-1)(1+\frac{1}{2}c-\frac{1-2\lambda}{2}d)$$

we have $\lambda(1-\lambda)$ is increasing with λ, Δ is increasing with λ, therefore $m(\lambda - p_1^*)$ is decreasing with m.

Then we proof mp_1^* is increasing with m:

$$mp_1^* = m\lambda - m(\lambda - p_1^*)$$

$m\lambda = \frac{m}{m+1}$ is increasing with m, and $m(\lambda - p_1^*)$ is decreasing with m, therefore we have mp_1^* is increasing with m.

Then we proof mp_k^* is increasing with m:

$$mp_k = 1 - \lambda + 2d\lambda(1-\lambda)\frac{2 + cl + (1-2\lambda)dl}{4 - c^2l + (1-2\lambda)^2d^2l}$$

$$\begin{aligned}\frac{\partial mp_k}{\partial \lambda} &= -1 + 2d\frac{(4 - c^2l + (1-2\lambda)^2d^2l)((1-2\lambda)(2 + cl + (1-2\lambda)dl) - 2\lambda(1-\lambda)dl)}{(4 - c^2l + (1-2)^2d^2l)^2} \\ &+ 2d\frac{4\lambda(1-\lambda)(2 + cl + (1-2\lambda)dl)(1-2\lambda)d^2l}{(4 - c^2l + (1-2\lambda)^2d^2l)^2}\end{aligned}$$

Denote $\square = 4 - c^2l + d^2l$, $1 - 2\lambda = r$, $\lambda(1-\lambda) = t$,

$$\begin{aligned}\frac{\partial mp_k}{\partial \lambda} &= -1 + 2d\frac{r(2 + cl + rdl)\square - 2tdl(\square - 4td^2l)}{(\square - 4td^2l)^2} \\ &= \frac{2dr(2 + cl + rdl)\square - 4td^2l(\square - 4td^2l) - (\square - 4td^2l)^2}{(\square - 4td^2l)^2} \\ &= \frac{\square}{(\square - 4td^2l)^2}(2dr(2 + cl + rdl) + 4td^2l - \square)\end{aligned}$$

Consider

$$
\begin{aligned}
F(\lambda) &= r(2 + cl + rdl) + 2tdl \\
&= (1-2\lambda)(2 + cl + (1-2\lambda)dl + 2\lambda(1-\lambda)dl) \\
&= (2 + cl + dl) - 2\lambda((2+cl) + (1-\lambda)dl) \\
&= (2 + cl + dl) - 2\lambda(2 + cl + dl - dl\lambda)
\end{aligned}
$$

$F(\lambda)$ is decreasing with λ, when $\lambda = 0$, $F_{max} = 2 + cl + dl$; when $\lambda = \frac{1}{2}$, $F_{min} = \frac{1}{2}dl$; we have $2dF_{max} < \square$ all the time, *i.e.* $\frac{\partial mp_k}{\partial \lambda} < 0$, mp_k will always increasing with m.

Finally we proof $m(p_k^* - \lambda)$ is deceasing with m:

$$m(p_k^* - \lambda) = mp_k - m\lambda$$

$$\frac{\partial m(p_k^* - \lambda)}{\partial \lambda} = \frac{\square}{(\square - 4td^2l)^2}(2dF(\lambda) - \square) + 1$$

$$\square(2dF(\lambda) - \square) + (\square - 4t^2d^2l)^2 = 2d\square(F(\lambda) - 4t^2dl) + (4t^2d^2l)^2$$

$F(\lambda) - 4t^2dl$ is decreasing with λ, when $\lambda = \frac{1}{2}$, $F(\lambda) - 4t^2dl = \frac{1}{2}dl - \frac{1}{4}dl > 0$, therefore $\frac{\partial m(p_k^* - \lambda)}{\partial \lambda} > 0$.

F Poof for Theorem 5

$$p_k^M = \frac{1}{2} + \frac{1}{4}d\Delta_1(\beta + \frac{1}{2}c(n-1))$$

$$2p_k^2 = \frac{2}{3} + \frac{2}{9}d\Delta_2(\beta + \frac{c(n-1)}{2} + \frac{1}{6}d(n-1))$$

$$2p_k^2 - p_k^M < \frac{1}{6} - \Delta_1 d(\frac{1}{36}\beta + \frac{1}{72}c(n-1) - \frac{1}{27}d(n-1))$$

Next we proof if $\beta > \frac{5}{3}\sqrt{n-1}$, we have:

$$\frac{1}{2}\Delta_1^{-1} < \frac{1}{12}d\beta + \frac{1}{24}d^2(n-1) - \frac{1}{9}d^2(n-1)$$

$$\iff \frac{1}{2} < \frac{1}{18}d^2(n-1) + \frac{1}{12}d\beta$$

$$\iff \beta > \frac{18 - 2d^2(n-1)}{3d}$$

Recall assumption **??**, we have $d^2(n-1) < c^2(n-1) < 4$, therefore

$$\frac{18 - 2d^2(n-1)}{3d} < \frac{5}{3}\sqrt{n-1} < \beta$$

G Social welfare

Let N_i denote the mass of consumers buying device $i = 1, .., n$ and denote $\tilde{p}_i = p_i - \sum_{j \neq i} \alpha_{ij} N_j = \sum_{k=1}^{m} p_i^k - \sum_{j \neq i} \alpha_{ij} N_j$ the "externality-adjusted" price of device i. Recall that $N_i = a_i - \tilde{p}_i$.

Aggregate consumer surplus is given by

$$\begin{aligned} CS &= \sum_i \int_{\tilde{p}_i}^{a_i} (a_i^\theta - \tilde{p}_i) da_i^\theta \\ &= \sum_i \left(\int_{\tilde{p}_i}^{a_i} a_i^\theta da_i^\theta - N_i \tilde{p}_i \right). \end{aligned}$$

Since

$$\int_{\tilde{p}_i}^{a_i} a_i^\theta da_i^\theta = \frac{1}{2}(a_i^2 - \tilde{p}_i^2) = \frac{1}{2}(a_i - \tilde{p}_i)(a_i + \tilde{p}_i) = \frac{N_i}{2}(2a_i - N_i)$$

we get

$$CS = \sum_i (a_i N_i - \frac{N_i^2}{2} - N_i \tilde{p}_i) = \sum_i (a_i N_i - \frac{N_i^2}{2} - N_i p_i) + \sum_i \sum_{j \neq i} \alpha_{ij} N_i N_j.$$

Therefore, social welfare is given by

$$W = \sum_i (a_i N_i - \frac{N_i^2}{2}) + \sum_i \sum_{j \neq i} \alpha_{ij} N_i N_j.$$

Differentiating with respect to N_i implies that at the social optimum, it must hold that

$$a_i - N_i + \sum_{j \neq i} (\alpha_{ij} + \alpha_{ji}) N_j = 0 \tag{G.1}$$

Denote p_i^W the socially optimal price of device i and N_i^W the social optimal mass of users of device i. Combining (G.1) with $N_i^W = a_i - p_i^W + \sum_{j \neq i} \alpha_{ij} N_j^W$ yields

$$p_i^W = -\sum_{j \neq i} \alpha_{ji} N_j^W. \tag{G.2}$$

The socially optimal price is therefore equal to the marginal cost (normalized to 0) minus the value created for the other users.

Denoting $\mathbf{p}^W = (p_1^W, p_2^W, ..., p_n^W)'$ and $\mathbf{N}^W = (N_1^W, N_2^W, ..., N_n^W)'$, we can rewrite the system of equations (G.2) as

$$\mathbf{p^W} = -\mathbf{A}'\mathbf{N}^W$$

Combining this with

$$\mathbf{N}^W = (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{a} - \mathbf{p^W})$$

leads to

$$\mathbf{p^W} = -\mathbf{A}'(\mathbf{I} - \mathbf{A})^{-1}(\mathbf{a} - \mathbf{p^W})$$

that is,

$$[\mathbf{I} - \mathbf{A}'(\mathbf{I} - \mathbf{A})^{-1}]\mathbf{p^W} = -\mathbf{A}'(\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}.$$

Thus,

$$\begin{aligned}
\mathbf{p^W} &= -[\mathbf{I} - \mathbf{A}'(\mathbf{I} - \mathbf{A})^{-1}]^{-1}\mathbf{A}'(\mathbf{I} - \mathbf{A})^{-1}\mathbf{A} \\
&= -[\mathbf{I} - \mathbf{A}'(\mathbf{I} - \mathbf{A})^{-1}]^{-1}[(\mathbf{I} - \mathbf{A})\mathbf{A}'^{-1}]^{-1}\mathbf{a} \\
&= -\left[[(\mathbf{I} - \mathbf{A})\mathbf{A}'^{-1}][\mathbf{I} - \mathbf{A}'(\mathbf{I} - \mathbf{A})^{-1}]\right]^{-1}\mathbf{a} \\
&= -[(\mathbf{I} - \mathbf{A})\mathbf{A}'^{-1} - \mathbf{I}]^{-1}\mathbf{a} \\
&= -[(\mathbf{I} - \mathbf{A})\mathbf{A}'^{-1} - \mathbf{A}'\mathbf{A}'^{-1}]^{-1}\mathbf{a} \\
&= -\mathbf{A}'[\mathbf{I} - (\mathbf{A} + \mathbf{A})']^{-1}\mathbf{a}
\end{aligned}$$

Assumption 1 $\lambda_{max}(\mathbf{A} + \mathbf{A}') < 1$, *where* λ_{max} *is the largest eigenvalue.*

Theorem 6 *Under assumption 1, the welfare-maximizing prices are given by*

$$\mathbf{p^W} = -\mathbf{A}'[\mathbf{I} - (\mathbf{A} + \mathbf{A})']^{-1}\mathbf{a} \tag{G.3}$$

In order to understand the welfare-maximizing prices in (G.3), let us compare it with the monopoly prices $\mathbf{p}^M$ in (5). First, in the absence of the externalities, the baseline prices of $\mathbf{p}^W$ are equal to the marginal costs, which we normalized to zero. Second, in the presence of externalities, the monopoly faces a trade-off between surplus extraction and internalization of externality which is captured by the term $(\mathbf{A} - \mathbf{A}')$ multiplied by the vector of centrality measure. In contrast, the social planner cares only about the internalization of externality as extracting surplus is a pure transfer: only $-\mathbf{A}'$ is multiplied by a vector of centrality measure. Hence, the social planner subsidizes all devices: all their prices are below the corresponding marginal costs. Last, the matrix to compute the centrality measure used by the social planner is different from the one used by a monopoly platform. While the social planner cares about the social marginal surplus, a monopoly platform cares about its marginal profit. The social marginal surplus can be expressed by rewritting (G.1) in a matrix form as

$$\underbrace{\mathbf{A} - [\mathbf{I} - (\mathbf{A} + \mathbf{A}')]\mathbf{N}}_{\text{social marginal surplus}} = \mathbf{0},$$

while the marginal profit is obtained from the first-order condition of the monopolist's profit, $\mathbf{\Pi}^M$=$[\mathbf{A} - (\mathbf{I} - \mathbf{A})\mathbf{N}]'\mathbf{N}$, with respect to $\mathbf{N}$:

$$\underbrace{\mathbf{A} - 2\left[\mathbf{I} - \frac{(\mathbf{A} + \mathbf{A}')}{2}\right]\mathbf{N}}_{\text{marginal profit}} = 0$$

Comparing the social marginal surplus and the marginal profit shows why the matrix to compute the centrality measure used by the social planner is different from the one of the monopolist.

We adapt the previous definitions introduced in the monopoly case as follows:

Definition 3 *The $n \times 1$ vector $[\mathbf{I} - [\mathbf{A} + \mathbf{A}']]^{-1}\,\mathbf{a} \equiv \mathbf{c}^{KB,w}$ measures each device's Katz-Bonacich centrality in the network $(\mathbf{A} + \mathbf{A}')$.*

Definition 4 $\mathbf{A}'\mathbf{c}^{KB,w}$ *measures out-degree Katz-Bonacich centrality of n devices.*

Then, $\mathbf{p}^W$ is given by

$$\mathbf{p}^W = -\mathbf{A}'\mathbf{c}^{KB,w}$$

Consider now the star network in (C):

$$\mathbf{A} = \begin{pmatrix} 0 & \boldsymbol{\eta}' \\ \boldsymbol{\mu} & \mathbf{0} \end{pmatrix}; \mathbf{A}' = \begin{pmatrix} 0 & \boldsymbol{\mu}' \\ \boldsymbol{\eta} & \mathbf{0} \end{pmatrix}.$$

Each device's KB centrality of $\mathbf{A} + \mathbf{A}'$ is given as follows:

$$c_1^{KB,W} = \frac{1 + c(n-1)}{1 - c^2(n-1)}$$

$$c_k^{KB,W} = \frac{1 + c}{1 - c^2(n-1)}, k = 2, 3, \cdots, n$$

The welfare-maximizing prices are given as follows:

$$p_1^W = -\sum_{k=2}^{n} \eta_k c_k^{KB,W}$$

$$p_k^W = -\mu_k c_1^{KB,W}, k = 2, 3, \cdots, n$$