Focality Dynamics and Platform Competition

Ma Hongxia*

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Abstract

We study how the evolution of consumer beliefs over time shapes dynamic competition between two vertically differentiated platforms in the presence of network effects, using the notion of partial focality. We develop a tractable model of focality updating based on past market outcomes, with exponential weighting governed by a persistence parameter. We capture asymmetries in platforms' focality through both the initial focality level and the persistence of the focality advantage. We show how focality dynamics affect market efficiency, consumer surplus, and comparative statics with respect to platforms' discount factors. Finally, when platforms can make initial-stage marketing investments that shift either the initial focality level or the persistence of focality dynamics, we show that the high-quality platform always has a stronger incentive to invest than the low-quality platform.

1 Introduction

In markets with network effects, consumers face a coordination problem because each consumer prefers to join the platform that other consumers are expected to join. When consumers coordinate on one platform rather than another for given prices, that platform gains a competitive advantage that allows it to charge a higher price. We refer to such a platform as *focal*. The strength of focality can vary. The higher its focality, the stronger this advantage. A more focal platform therefore faces weaker coordination frictions, whereas a less focal platform may be disadvantaged simply because consumers do not expect it to attract others.

In many markets, focality is history dependent. A platform that wins today is more likely to be expected to attract all consumers in the next period, whereas a platform that loses ground becomes less attractive over time. This dynamic creates a natural source of persistence in market leadership. In particular, an incumbent may remain difficult to displace even when an entrant offers higher intrinsic quality, simply because the incumbent begins with a more favorable focal position.

A firm's focality process has two dimensions. The first is its current level, which captures the platform's current expectation advantage. The second is its persistence, which determines how strongly focality responds to new market outcomes. When focality is more persistent, current beliefs react less to recent

*Department of Economics, National University of Singapore, Singapore. Email: hongxia_ma@u.nus.edu.

outcomes and reflect past success more heavily. Platforms may therefore differ not only in how focal they are at a point in time, but also in how durable that focality is. A firm's focality may also be more persistent than its rival's, so that even after a loss, its focality declines by less. Such differences may arise from brand recognition, consumer loyalty, or reputational capital.

Platforms therefore have incentives to invest in improving focality. The first channel is subsidizing early participation. A platform may subsidize participation in early periods in order to improve future focality and recoup those losses later. The profitability of such a strategy depends on the law of motion of focality. As platforms become more patient, the incentive of the high-quality entrant to invest in winning today becomes stronger, but so does the incentive of the lower-quality incumbent to defend its current position. Whether a higher discount factor improves market efficiency is therefore not immediate. The answer depends on focality dynamics.

The second channel is direct investment in focality through advertising, launch campaigns, or related forms of marketing that influence consumer coordination. These investments can raise the current level of focality by convincing consumers that more others will join the platform. They can also increase the persistence of focality by making consumers believe that the platform's appeal will last over time, for example by building brand recognition.

In this paper, we study how focality dynamics shape price competition between two vertically differentiated platforms. We develop a tractable model of the focality updating process, in which platforms' focality levels evolve according to an exponentially weighted updating rule. The framework allows platforms to differ along both dimensions described above: the current level of focality and the persistence of the focality process. A platform with more persistent focality retains more of the advantage generated by past success and suffers less from an unfavorable current outcome.

The paper addresses three questions. First, how do focality dynamics affect market selection, efficiency, and consumer surplus? Second, how do the effects of firms' discount factors depend on the focality updating process? Third, when platforms can invest to affect focality, which platform has the stronger incentive to do so?

We begin with an exogenous focality process. In the benchmark case of symmetric persistence, we characterize the equilibrium cutoff focality level above which the lower-quality platform wins. This cutoff captures the focality advantage required for the lower-quality platform to offset its quality disadvantage. We show that greater persistence lowers this cutoff. The intuition is straightforward: when focality is more persistent, an early expectation advantage is harder to overturn. As a result, quality has less influence on market selection, the lower-quality platform is more likely to survive, and allocative efficiency declines. Consumer surplus also falls because the higher-quality platform has weaker incentives to subsidize aggressively when the rival's focality advantage is more durable. This contrasts with the common view that higher persistence raises consumer surplus by increasing the value of winning the market today and hence intensifying dynamic price competition. We show that this common view does not hold in general.

We then allow persistence to differ across platforms. This extension introduces a distinct force beyond the overall degree of persistence, namely relative persistence. A platform whose focality is more persistent has an advantage in focality updating. When the high-quality platform has a relative persistence advantage, the dynamics work in its favor, and a higher discount factor improves market efficiency. By contrast, when the low-quality platform has a persistence advantage, the effect of dynamics on efficiency is positive only if the quality gap is sufficiently large.

Finally, we endogenize focality by allowing platforms to make initial-stage marketing investments. We consider two channels. Under a level effect, marketing shifts the initial focality level. Under a persistence effect, marketing changes the persistence of focality. In both cases, the higher-quality platform has the stronger incentive to invest. The reason is that such investments are more valuable for the higher-quality platform in relaxing the coordination disadvantage created by an unfavorable focal position. Thus, whether marketing affects the level of focality or the persistence of the focality process, the direction of the investment incentive is the same.

The remainder of the paper is organized as follows. Section 2 presents the model and studies the two-period case. Section 3 studies the infinite-horizon benchmark with symmetric persistence. Section 4 extends the analysis to asymmetric persistence. Section 5 studies marketing investments that affect focality. Section 6 concludes.

Related Literature. We develop a tractable model of focality dynamics based on exponentially weighted updating, which yields sharp comparative statics in an infinite-horizon environment.

Our paper is related to the theoretical literature on coordination in platform competition. Caillaud and Jullien (2001) and Caillaud and Jullien (2003) introduce the idea of favorable beliefs, which is closely related to what we call focality. Hagiu (2006) and Jullien (2011) extend this insight to alternative pricing environments. Halaburda and Yehezkel (2013) study platform competition under asymmetric information, while Halaburda and Yehezkel (2016) analyze coordination bias in two-sided markets. Most closely related is Halaburda and Yehezkel (2019), which introduces the notion of partial focality. Across this literature, focality shapes the nature of platform competition.

In dynamic environments, however, focality may itself be shaped by market outcomes. Consumers may use a platform's market history to form beliefs, assigning greater focality to the platform that dominated in the past. In this spirit, Argenziano and Gilboa (2012) study how history can serve as a coordination device, while Biglaiser and Crémer (2020) analyze dynamic platform competition with incumbency advantage and heterogeneous consumers.

We contribute to this literature by providing a tractable model in which consumer beliefs evolve with market history through exponentially weighted updating. This also connects our paper to a broader literature on learning and belief formation. Muth (1960) provides an early analysis of exponentially weighted forecasting rules. In the adaptive-learning literature, Marcet and Sargent (1989) study recursive belief updating and the convergence of least-squares learning in self-referential stochastic models. In games, Camerer and Ho (1999) propose experience-weighted attraction learning, a recency-sensitive updating framework that combines reinforcement and belief-based learning.

Our framework can also be viewed as a general model of focality dynamics. Halaburda, Jullien, and

Yehezkel (2020) analyze a special case in which consumers assign full focality to the platform that dominated the market in the previous period.

2 The Model

Consider a homogeneous consumer population of size 1 and two competing platforms, which we refer to as the incumbent I and the entrant E. Both platforms face the same production cost, normalized to 0. There are T periods, $t = 0, 1, 2, \cdots, T-1$, where T may be finite or infinite.

Each platform i offers consumers a good or service generating stand-alone value $q_i > 0$, which we refer to as *quality*. Without loss of generality, assume that platform I is the lower-quality platform, so that $q_I < q_E$. Denote $q = q_I - q_E$ as the quality gap.

Additionally, consumers benefit from network effects. A consumer's utility from buying the good or service of, and thus joining, platform i is $q_i + \beta n_i - p_i$, where n_i is the measure of other consumers who have joined platform i, β denotes the strength of network effects, and p_i is the price charged by platform i.

In each period t, the platforms first simultaneously set their prices, $p_I(t)$ and $p_E(t)$. A negative price is interpreted as a price below cost. Consumers then observe the prices and simultaneously and non-cooperatively choose whether to join platform I, platform E, or neither. In the latter case, utility is zero. The two platforms operate for T periods and discount future profits by δ, where $0 < \delta < 1$.

There are no switching costs, so consumers' current payoffs are not directly affected by past or future periods. Path dependence arises solely from belief formation. Moreover, since consumers are homogeneous, in equilibrium they all join the same platform, leaving the rival platform empty. An implicit assumption of the model is that a firm remains active even when its equilibrium continuation value is zero. This is consistent with the absence of any fixed operating cost and implies that each firm always faces some competitive pressure.

Competition in an environment with network effects often results in multiple equilibria, and that is also the case here. Consider the allocation of consumers that emerges for given prices. There is an equilibrium in which all consumers join platform I if

$$q_I - p_I(t) + \beta \geq q_E - p_E(t),$$

and an equilibrium in which all consumers join platform E if

$$q_I - p_I(t) \leq q_E - p_E(t) + \beta.$$

These two conditions overlap. In particular, when $q_E - q_I - \beta < p_E(t) - p_I(t) < q_E - q_I + \beta$, there are two possible allocations: either all consumers join I or all join E. This in turn leads to multiple equilibria in platforms' pricing. We rely on the notion of partial focality to select equilibrium.

Caillaud and Jullien (2001, 2003) suggest that consumer beliefs can favor one of the platforms, that is, one of the two platforms is focal. Halaburda and Yehezkel (2019) extend this idea to partial focality. When prices in period t are compatible with multiple allocations of consumers, platform I is focal to degree α_t ($0 \leq \alpha_t \leq 1$), and agents join platform I if

$$q_I - p_I(t) + \alpha_t \beta \geq q_E - p_E(t) + (1 - \alpha_t)\beta, \tag{1}$$

and join platform E otherwise. Equivalently, platform E's partial focality level at t is $1 - \alpha_t$. When $\alpha_t > \frac{1}{2}$, platform I has a relative focality advantage that allows it to set a higher price and still win the market than the other platform.

Firms' focality levels in period t are determined by the history of market outcomes. A platform's focality weakly increases when it wins a period and weakly decreases when it loses. Let $o_t \in \{I, E\}$ denote the winner in period t, and let $x_t = \mathbf{1}(o_t = I)$. Let $H = \{o_\tau\}_{\tau=0}^{t-1}$ denote the history at the start of period t. The focality level in period t is given by

$$\alpha_t = F(H),$$

for some mapping $F : \{I, E\}^t \to [0, 1]$. Moreover, we refer to a history path as a sequence $(H_t)_{t>0}$ with $H_{t+1} = (H_t, o_t)$ for every $t \geq 0$.

For tractability, we assume that α_t is a sufficient statistic for the payoff-relevant history. Thus, α_{t+1} depends on past outcomes only through α_t and the current-period outcome. A convenient specification is the following exponentially weighted updating rule.

Assumption 1 (EWMA focality updating process) *Focality evolves according to an exponentially weighted moving-average (EWMA) rule:*

$$\alpha_{t+1} = (1 - \rho_t)x_t + \rho_t \alpha_t, \qquad \rho_t \in (0, 1).$$

The parameter ρ_t governs the persistence of focality following the period-t outcome: a higher ρ_t implies a more persistent response and slower adjustment.

For stability, we assume that there exists $\bar{\rho} < 1$ such that $\rho_t \leq \bar{\rho} < 1$ for all t. This guarantees that the updating rule is a contraction and ensures convergence of the focality process.

This updating rule captures the idea that the influence of past outcomes fades over time. Iterating the updating rule yields the following recency-weighted representation of focality. For any $t \geq 1$,

$$\alpha_t = \left(\prod_{s=0}^{t-1} \rho_s \right) \alpha_0 + \sum_{n=1}^{t} \left[(1 - \rho_{t-n}) \prod_{s=t-n+1}^{t-1} \rho_s \right] x_{t-n}.$$

Hence, α_t is a convex combination of the initial focality level α_0 and past outcomes $\{x_{t-1}, \ldots, x_0\}$. The weight on the outcome realized in period $t - n$ is

$$w_{t,n} = (1 - \rho_{t-n}) \prod_{s=t-n+1}^{t-1} \rho_s.$$

Thus, the effect of a given market outcome on future focality is geometrically discounted over time: once an outcome is realized, its contribution to α_t shrinks as it recedes into the past, so sufficiently distant outcomes have only negligible influence on current focality.

In period 0, we assume that an initial focality level $\alpha_0 \in [0, 1]$ is arbitrarily given. For any history H, let α^H denote the focality level induced by H. Moreover, the realization of x_t induces two affine maps

$$f_I(\alpha_t) = (1 - \rho_t) + \rho_t\alpha_t, \quad f_E(\alpha_t) = \rho_t\alpha_t,$$

where $f_I(\alpha_t)$ and $f_E(\alpha_t)$ denote platform I's focality level after platform I wins and after platform E wins in t, respectively.

In period t, if platform I loses, then its focality level in the next period falls by

$$(1 - \rho_t)\alpha_t$$

relative to its period-t level. Thus, a loss causes a proportional decline in platform I's current focality. Therefore, ρ_t conditional on platform I losing in period t, captures the resilience of platform I's focality, and we denote it by ρ_I. A higher ρ_I means that a larger fraction of platform I's focality will be retained after a loss.

On the other hand, if platform I wins in period t, then platform E's focality decreases by

$$(1 - \rho_t)(1 - \alpha_t),$$

relative to its period-t level $1 - \alpha_t$. Therefore, ρ_t, conditional on E loses in period t, measures the resilience of E's focality, and we denote it as ρ_E.

Thus, by allowing the persistence parameter to depend on which platform loses in period t, i.e., $\rho_t \in \{\rho_I, \rho_E\}$, we capture asymmetry in the persistence of focality advantage across the two platforms, and hence asymmetry in focality updating. It follows that:

1. If $\rho_I > \rho_E$, then platform I's focality is more resilient: when platform I loses, its focality falls by a smaller proportion of its current level than platform E's focality falls when platform E loses. Equivalently, when platform I wins, it acquires a larger proportion of platform E's focality than platform E acquires of platform I's focality when platform E wins.

2. If $\rho_I < \rho_E$, then platform E's focality is more resilient. Relative to platform E, platform I loses more from a loss and gains less from a win, in proportional terms.

Platforms can therefore differ in focality process along two dimensions: the current level of focality, captured by $(\alpha_t, 1 - \alpha_t)$, and the persistence of the focality, captured by (ρ_I, ρ_E), which govern the proportional response in the current focality level to wins and losses.

Holding the persistence parameters fixed, a higher focality level implies a smaller increase in focality following a win. Thus, the marginal return to winning decreases with the platform's current focality level:

along a path of successive wins, the marginal gain from additional win is decreasing. Equivalently, a higher focality level implies a larger decline following a loss. A platform with higher focality is more vulnerable to losses.

We first study the case with an exogenous focality updating process. We start with a simple two-period model and then extend the analysis to the infinite-horizon case.

2.1 Two-period model

We consider $T = 2$ and show that there exists a unique subgame perfect Nash equilibrium. In equilibrium, one platform keeps winning for both periods, and the low-quality platform wins if and only if its initial focality level is higher than than a threshold value, which is determined by the focality updating as well as platforms' discount factor.

To solve the model, we start with the second period, $t = 1$. Conditional on platform I winning in period 0, platform I's focality level in this period is $f_i(\alpha_0)$. Because in this period there is no future, the subgame is identical to the static case, in which neither platform is willing to set a negative price. Hence, if platform I wins, its equilibrium profit is

$$q_I - q_E + \beta(2f_i(\alpha_0) - 1).$$

We now turn to period 0. When setting prices in the first period, each platform takes into account that winning the market in that period affects focality in the next period, and hence future profit. In particular, each platform is willing to offer a subsidy of up to the difference between its discounted continuation profit following a win in period 0 and its discounted continuation profit following a loss.

Consider first an equilibrium in which platform I wins in period 0. The lowest price that platform E is willing to charge is

$$p_E(0) = -\delta\Big(q_E - q_I - \beta(2f_E(\alpha_0) - 1)\Big).$$

To win the market, platform I must then set

$$p_I(0) = q_I - q_E + \beta(2\alpha_0 - 1) + p_E(0).$$

If it wins in period 0, it also wins in period 1, so its discounted total profit is

$$q_I - q_E + \beta(2\alpha_0 - 1) + 2\delta\Big(q_I - q_E + \beta(f_I(\alpha_0) + f_E(\alpha_0) - 1)\Big).$$

Likewise, in an equilibrium where platform E wins in both periods, platform E's discounted total profit is given by the negative of the above expression.

Therefore, in equilibrium, platform I wins in all periods if and only if

$$\alpha_0 \geq \hat{\alpha}(q),$$

and platform E wins in all periods otherwise, where $q = q_I - q_E$. For notational simplicity, write $\hat{\alpha} \equiv \hat{\alpha}(q)$.

The threshold $\hat{\alpha}$ satisfies

$$\hat{\alpha} = -\frac{(1+2\delta)(q-\beta)}{2\beta} - \delta(f_I(\hat{\alpha}) + f_E(\hat{\alpha}) - 1).$$

A higher threshold $\hat{\alpha}$ means that platform I, the low-quality platform, requires a higher initial focality level to win. Hence, the high-quality platform wins for a larger set of initial conditions, implying higher efficiency.

Clearly, $\hat{\alpha}$ is decreasing in q, as well as in $f_I(\alpha_0)$ and $f_E(\alpha_0)$, which are determined by the focality updating process. Yet, platform I may win and so move ahead even if it starts with a relatively low initial focality level, when it has an advantage in focality updating process that its focality is relatively more persistent.

Meanwhile, as δ increases, greater weight is placed on the focality dynamics $f_I(\alpha_0)$ and $f_E(\alpha_0)$, which may increase or decrease $\hat{\alpha}$. We have:

$$\frac{\partial\hat{\alpha}}{\partial\delta} > 0 \quad \Leftrightarrow \quad \hat{\alpha}_0 - f_E(\hat{\alpha}_0) > f_I(\hat{\alpha}_0) - \hat{\alpha}_0,$$

where $\hat{\alpha}_0 \equiv \hat{\alpha}(q|\delta = 0)$. This result is consistent with Halaburda and Yehezkel (2019). The intuition is as follows. Starting from the static benchmark, suppose $\alpha_0 = \hat{\alpha}_0$ such that we have reached balance between platforms' focality advantage and quality advantage for the static case. When dynamics is introduced, the way focality updates in the second period shifts this balance point. If platform I's focality falls more after a loss than it rises after a win, then focality dynamics work against platform I. As δ increases, the dynamics make it harder for platform I to win. As a result, $\hat{\alpha}$ increases and efficiency rises. Conversely, if platform I's focality rises more after a win than it falls after a loss, then the dynamics works in its favor. As δ increases, $\hat{\alpha}$ decreases and efficiency falls.

Our specification of the focality updating process enables us to derive additional results. Under the EWMA updating rule, the condition becomes

$$(1-\rho_I)\,\hat{\alpha}_0 > (1-\rho_E)(1-\hat{\alpha}_0).$$

It holds when

$$q_E - q_I > \beta\frac{\rho_I - \rho_E}{2 - \rho_I - \rho_E}.$$

Given (ρ_I, ρ_E), we have:

1. if $\rho_I \leq \rho_E$, i.e., the high-quality platform E's focality is weakly more persistent, then a higher δ always increases efficiency.

2. if $\rho_I > \rho_E$, so that platform I's focality is more persistent, then a higher δ increases efficiency only if the quality gap is sufficiently large. The intuition is that, when the quality gap is large, the focality level required by the low-quality platform to offset its quality disadvantage in the static benchmark is higher. Once dynamics are introduced, its focality rises by less after a win and falls by more after a loss. As a result, a sufficiently large quality gap makes the overall dynamics favor the high-quality platform.

The two-period case provides a useful benchmark and helps build intuition. In the remainder of the paper, we focus on the infinite-horizon model. Our focus is on Markovian strategies and a Markov-perfect equilibrium (MPE). Among MPEs, we rule out equilibria which are supported by the losing firm pricing in such a way that its value would be negative if it instead won consumers at the prices charged.

3 Benchmark: symmetric focality updating

We first consider a benchmark case in which platforms have symmetric focality updating process: $\rho_I = \rho_E = \rho$. The common persistence ρ captures the persistence of the history itself. A higher ρ implies greater persistence of history and slower adjustment to the current outcome, regardless of which platform wins in the current period.

With constant ρ, after a history H of length t, platform I's focality level becomes

$$\alpha^H = \rho^t \alpha_0 \; + \; (1-\rho)\sum_{s=1}^{t} \rho^{t-s} x_s,$$

with the weight on each period outcome strictly decreasing in its distance from the current period.

The relevant state at the start of any period is firm I's focality level α_t. The corresponding value functions for I and E (i.e., the present discounted values (PDVs) of future profit flows starting with the current period when firms follow their equilibrium pricing strategies in every period) are denoted by $V_I(\alpha_t)$ and $V_E(\alpha_t)$.

In period t, in order to win the current period, each firm will be willing to offer a subsidy equal to the difference in the discounted value of its future profits if it were to win the current period and its future profits if it were to lose the current period. First, consider an equilibrium in which firm I wins in period t. Platform I also wins in period $t+1$, as winning increases its focality level. The lowest price platform E is willing to set in period t is $p_E(t) = -\delta V_E(f_E(\alpha_t))$. In order to win, platform I must set $p_I(t) = q + \beta(2\alpha_t - 1) - \delta V_E(f_E(\alpha_t))$ and therefore obtains total profit

$$V_I(\alpha_t) = q + \beta(2\alpha_t - 1) + \delta V_I(f_I(\alpha_t)) - \delta V_E(f_E(\alpha_t)).$$

Such an equilibrium, in which platform I wins in every period, exists if this expression is non-negative. Likewise, the equilibrium in which platform E wins in every period exists if platform E's total profit:

$$V_E(\alpha_t) = -q - \beta(2\alpha_t - 1) - \delta V_I(f_I(\alpha_t)) + \delta V_E(f_E(\alpha_t)).$$

is non-negative.

Combining the two conditions under which the respective equilibria exist, for any current focality state α_t, at most one of $V_I(\alpha_t)$ and $V_E(\alpha_t)$ can be strictly positive. Since $V_I(\alpha_t)$ is increasing in α_t, while $V_E(\alpha_t)$ is decreasing in α_t, for a given quality gap q there exists a threshold $\hat{\alpha}(q)$ such that

$$V_I(\alpha_t) > 0 \iff \alpha_t > \hat{\alpha}(q),$$

$$V_E(\alpha_t) > 0 \iff \alpha_t < \hat{\alpha}(q).$$

Moreover, whenever the value functions are finite,

$$V_I(\hat{\alpha}(q)) = V_E(\hat{\alpha}(q)) = 0.$$

Thus, $\hat{\alpha}(q)$ is the threshold focality level above which platform I wins. Since $q < 0$, this cutoff satisfies $\hat{\alpha}(q) > \frac{1}{2}$ and is decreasing in q. In particular, in equilibrium, platform I wins in every period if and only if $\alpha_0 > \hat{\alpha}(q)$, with prices given by, for $t \geq 0$,

$$p_I(t) = q + \beta(2\alpha^{I^t} - 1) - \delta V_E(\alpha^{I^t E}), \qquad p_E(t) = -\delta V_E(\alpha^{I^t E}).$$

Platform E subsidizes in period t if and only if $V_E(\alpha^{I^t E}) > 0$. Since $V_E(\alpha^{I^t E})$ decreases with t, platform E's subsidizing amount decreases over time. Define

$$s^* = \max\{s \geq 0 : \alpha^{I^s E} < \hat{\alpha}\},$$

with the convention that $s^* = \infty$ if platform E subsidizes forever. If $s^* < \infty$, then platform E subsidizes only up to a finite period, s^*.

Our goal is to characterize $\hat{\alpha}(q)$. To do so, we take the quality gap q as given and expand the platforms' value functions starting from the initial focality level α_0. We denote $\hat{\alpha}(q)$ by $\hat{\alpha}$ for short.

Absorptions As shown above, each platform's value depends on the rival's next-period continuation value conditional on winning the current period. Starting from α_0, we therefore expand the recursion over all possible histories. For a given quality gap, it may happen that along some history one platform's focality becomes so strong that the rival cannot obtain a positive continuation value even if it wins in the current period. At that point, the rival no longer finds it profitable to set negative prices to compete, and the recursive expansion changes form. We refer to such histories as *absorbing*.

Consider a history H such that $V_I(\alpha^H) > 0$. Then

$$V_I(\alpha^H) = q + \beta(2\alpha^H - 1) + \delta V_I(f_I(\alpha^H)) - \delta V_E(f_E(\alpha^H)).$$

If

$$\alpha^H > \frac{\hat{\alpha}}{\rho}, \tag{2}$$

then $f_E(\alpha^H) > \hat{\alpha}$, and hence $V_E(f_E(\alpha^H)) = 0$. We refer to such a history as *I-absorbing*. At such a history, platform I wins, while platform E no longer has an incentive to set negative prices to compete. This remains true along the equilibrium continuation path thereafter.

The recursion therefore simplifies to

$$V_I(\alpha^H) = q + \beta(2\alpha^H - 1) + \delta V_I(f_I(\alpha^H)),$$

so the continuation value is given by

$$T_I(\alpha^H) := V_I(\alpha^H) = \frac{q+\beta}{1-\delta} - 2\beta\frac{1-\alpha^H}{1-\delta\rho}.$$

Relative to the non-absorbing case, this continuation payoff is strictly larger, because the term $-\delta V_E(f_E(\alpha^H))$ disappears once I-absorption is reached.

Holding fixed the other parameters, a higher $\hat{\alpha}$, which corresponds to a larger quality disadvantage for platform I, makes the condition for I-absorption harder to satisfy and therefore makes I-absorption less likely. Moreover, I-absorption is reachable under the given parameter values if there exists at least one I-absorbing history, that is, if

$$\hat{\alpha} < \rho.$$

When I-absorption is reachable, the equilibrium path $\{I^t\}$, along which platform I keeps winning, eventually enters the I-absorbing region. Equivalently, after some finite time platform E stops subsidizing, so $s^* < \infty$. Otherwise, platform E subsidizes forever.

Similarly, consider a history H such that

$$\alpha^H < 1 - \frac{1-\hat{\alpha}}{\rho}. \tag{3}$$

Then, in the continuation subgame, platform E keeps winning while platform I stops setting negative prices. We refer to such a history as *E-absorbing*. At such a history,

$$T_E(\alpha^H) := -V_E(\alpha^H) = \frac{q-\beta}{1-\delta} + \frac{2\beta\alpha^H}{1-\delta\rho}.$$

The continuation payoff for platform E at this history is positive and larger than in the non-E-absorbing case, which further lowers platform I's value in the recursive expansion.

Holding fixed the other parameters, a higher $\hat{\alpha}$ makes E-absorption easier to reach. Moreover, E-absorption is reachable when

$$\hat{\alpha} > 1 - \rho.$$

Taken together, the absorption regions determine which histories contribute to the value functions. Once a history enters an absorbing region, the continuation is pinned down: the same platform wins in every subsequent period along the equilibrium continuation path. Hence later histories beyond that point no longer generate additional variation in the value function.

Then we unpack the value function at α_0, taking q as given and working on the region $\alpha_0 > \hat{\alpha}$. Let $\mathcal{H}$ denote the set of finite histories along which absorption has not yet occurred, and let $\mathcal{H}^I$ (respectively,

$\mathcal{H}^E$) denote the set of finite histories that first reach I-absorption, (respectively, E-absorption). Then $V_I(\alpha_0)$ admits the following pruned sum representation:

$$V_I(\alpha_0) = \sum_{H\in\mathcal{H}} \delta^{|H|}(q + \beta(2\alpha^H - 1)) + \sum_{H\in\mathcal{H}^I} \delta^{|H|}T_I(\alpha^H) + \sum_{H\in\mathcal{H}^E} \delta^{|H|}T_E(\alpha^H).$$

This expression is piecewise affine in (α_0, q), with the relevant summation terms depending on whether, and at what point, each history path reaches absorption. Accordingly, the parameter space can be partitioned into three cases according to whether absorption is reachable.

(A) **No absorption.** If $\rho \le \hat{\alpha} \le 1 - \rho$, which requires $\rho < \frac{1}{2}$, then neither side absorption is reachable.

$$V_I(\alpha_0) = \frac{q}{1 - 2\delta} + \beta(2\alpha_0 - 1)\frac{1}{1 - 2\rho\delta},$$

for $\delta < \frac{1}{2}$. For $\delta \ge \frac{1}{2}$, the value function diverges. Then, the cutoff $\hat{\alpha}(q)$ above which $V_A(\alpha_0) > 0$ is given by

$$\hat{\alpha}(q) = \frac{1}{2} - \frac{q}{2\beta}\frac{1 - 2\rho\delta}{1 - 2\delta}, \qquad \text{for } \delta < \tfrac{1}{2},$$

Equivalently, the no-absorption case arises if and only if

$$|q| < \beta(1 - 2\rho)\frac{1 - 2\delta}{1 - 2\rho\delta},$$

that is, when the quality gap is sufficiently small and focality is not too persistent. When $\delta > \frac{1}{2}$, the cutoff $\hat{\alpha}(q)$ lies outside the no-absorption region for every q.

(B) **Both absorptions reachable.** If $1 - \rho < \hat{\alpha} < \rho$ (which requires $\rho > \frac{1}{2}$), then both absorptions are reachable. This case arises when the quality gap is not too large and focality is sufficiently persistent, and

$$V_I(\alpha_0) = \sum_{t=0}^{\infty} \delta^t(q + \beta(2\alpha^{I^t} - 1)) - \sum_{t=0}^{s^*} \delta^{t+1}V_E(\alpha^{I^tE}),$$

where $s^* < \infty$. Since the point at which absorption occurs varies across history paths, it is not possible to determine in closed form when each path reaches absorption. As a result, we cannot obtain an explicit expression for the value function or for the cutoff.

(C) **E-absorption only.** If $\hat{\alpha} > \max\{\rho, 1 - \rho\}$, then only E-absorption is reachable, and

$$V_I(\alpha_0) = \sum_{t=0}^{\infty} \delta^t(q + \beta(2\alpha^{I^t} - 1)) - \sum_{t=0}^{\infty} \delta^{t+1}V_E(\alpha^{I^tE}).$$

If we assume that each history $\{I^tE\}$ is E-absorbing, then we obtain a lower bound for $V_I(\alpha_0)$:

$$V_I'(\alpha_0) = \frac{q}{(1 - \delta)^2} + \frac{\beta}{(1 - \delta\rho)^2}\left(2\alpha_0 - 1 - \frac{\delta^2(1 - \rho)^2}{(1 - \delta)^2}\right),$$

Solving $V_I'(\alpha_0) = 0$ yields an explicit upper bound on $\hat{\alpha}(q)$:

$$\bar{\alpha}(q) = \frac{1}{2} + \frac{1}{2(1-\delta)^2}(\delta^2(1-\rho)^2 - \frac{q}{\beta}(1-\delta\rho)^2).$$

Moreover, $\hat{\alpha}(q) = \bar{\alpha}(q)$ when the above expression is larger than $1 - \rho + \rho^2$. Equivalently, when q is sufficiently negative, E-absorption becomes immediate after the first period in which platform E wins.

As q decreases further, $\hat{\alpha}(q)$ eventually reaches 1. The corresponding value of q defines the lower bound below which platform I cannot win for any initial focality level. Solving $\bar{\alpha}(q) = 1$ yields

$$\underline{q} = -\beta\frac{1 - 2\delta + \delta\rho}{1 - \delta\rho}.$$

Although absorptions prevent a fully explicit solution for $\hat{\alpha}$ in all parameter regions, the previous analysis identifies the key properties of the cutoff. In particular, we have characterized two polar cases in closed form and established some properties of firms' value functions. These results are sufficient both to summarize the equilibrium and to develop the comparative statics later.

Proposition 1 *(Equilibrium characterization). Fix (q_I, q_E, α_0). For each finite horizon T, the game admits a unique subgame perfect equilibrium. As $T \to \infty$, these equilibria converge to a limiting equilibrium. In the limiting equilibrium, platform I wins if and only if*

$$\alpha_0 \geq \hat{\alpha}(q).$$

The cutoff $\hat{\alpha}(q) \in [0, 1]$ is continuous, strictly decreasing in q. $\hat{\alpha}(q) > \frac{1}{2}$ for $q < 0$, and $\hat{\alpha}(q) < 1$ for $q > \underline{q}$.

This threshold identifies the minimum initial focality level advantage, $\hat{\alpha}$, that platform I needs to hold in order to win the market, given the quality gap q. The low-quality platform wins if and only if it starts with a sufficiently high focality level. Intuitively, if $q = 0$, then $\hat{\alpha} = \frac{1}{2}$, platforms are homogeneous. A slight degree of focality level advantage is enough for platform I to win the market and earn positive profits. As the quality gap enlarges, $\hat{\alpha}$ increases above $\frac{1}{2}$; platform I requires a stronger initial focality advantage to offset its quality disadvantage. When the quality gap is sufficiently large, platform I cannot win even if it starts with full focality. Moreover, the relationship between the quality gap q and the threshold focality level $\hat{\alpha}$ is nonlinear due to absorptions. As platform I's quality disadvantage grows, it requires a larger increase in its initial focality level to remain winning.

Fix q and set $\alpha_0 = \hat{\alpha}(q)$. We then study the effect of a higher δ. As in the two-period model, a higher δ strengthens both platforms' incentives to subsidize in early periods, and the net effect depends on the focality dynamics. Under common persistence, these dynamics work against the low-quality platform, and this can be seen from the absorption structure.

At α_0, there is weakly more E-absorption than I-absorption. Consider a pair of complementary histories (H_I, H_E), where H_I ends with platform I winning in the last period and H_E ends with platform E winning in the last period. The two histories represent the same ordered sequence of wins and losses, with platform identities swapped. Starting from $\alpha_0 = \hat{\alpha}$, For any pair of complementary histories (H_I, H_E),

$$H_I \text{ is I-absorbing} \implies H_E \text{ is E-absorbing},$$

whereas the converse implication needs not hold. Equivalently, along complementary histories, platform I's focality is more likely to cross the relevant cutoff in the unfavorable direction than platform E's.

Corollary 1 *In the symmetric benchmark, $\hat{\alpha}(q)$ is increasing in δ for $q < 0$, so efficiency level always increases as platforms become more forward-looking.*

When platforms' focality updating are symmetric, then in equilibrium where the low-quality platform I wins, the higher initial focality level transfers a disadvantage in focality dynamics: more vulnerable to losses than platform E. As δ increases, this unfavorable focality dynamics decreases I's expected profits. As a result, the low-quality platform requires a higher initial focality level to win. In other words, fixing the quality gap, there is a larger range of initial focality levels such that the high-quality platform wins the market, implying a higher level of efficiency.

In particular, when δ is high, even a small quality disadvantage must be offset by a sufficiently strong initial focality advantage. This also explains why, in the no-absorption case, no solution exists when $\delta > \frac{1}{2}$: when platforms are sufficiently forward-looking, future market capture matters so much that a quality-disadvantaged platform cannot win for any $\alpha_0 \in [\rho, 1-\rho]$.

3.1 Comparative statics in common persistence

We next study comparative statics with respect to the common persistence parameter ρ. Fix q and a baseline value of ρ, and set $\alpha_0 = \hat{\alpha}(q;\rho)$, so that the initial condition is exactly at the equilibrium threshold for which platform I wins. We then consider an increase in ρ, holding fixed this initial condition, and study how the equilibrium threshold, market efficiency, and consumer surplus respond.

A higher ρ makes focality more persistent for both platforms: past outcomes decay more slowly, so focality becomes less sensitive to the most recent outcome, regardless of which platform wins. In particular, platform I's focality falls less after a loss, but also rises less after a win. Hence, a higher ρ has two offsetting effects on platform I's focality: it cushions declines after losses, but mutes gains after wins.

Recall that $V_I(\alpha_0)$ admits a pruned-sum representation over histories:

$$V_I(\alpha_0) = \sum_{H_I \in \mathcal{H}^N} \delta^{|H_I|}(q + \beta(2\alpha^{H_I} - 1)) + \sum_{H_E \in \mathcal{H}^N} \delta^{|H_E|}(q + \beta(2\alpha^{H_E} - 1)),$$

where $\mathcal{H}^N$ denotes the set of histories that survive pruning and contribute to the value function. Specifically, $\mathcal{H}^N$ contains two types of histories: first is H_I that end with platform I winning in the last period and are not E-absorbed, meaning that no proper prefix of H_I has reached the E-absorbing region; second is H_E that end with platform E winning in the last period and are not I-absorbed, meaning that no proper prefix of H_E has reached the I-absorbing region. We first examine how an increase in ρ affects α^H along histories $H \in \mathcal{H}^N$.

Lemma 1 *Suppose $\alpha_0 = \hat{\alpha}$. Then, for any history H that contributes to the value function, α^H decreases in ρ if $\alpha^H > \alpha_0$, and increases in ρ if $\alpha^H < \alpha_0$.*

The lemma provides a partial comparative static with respect to ρ. Fix the initial value of ρ, and let $\alpha_0 = \hat{\alpha}$ be the corresponding equilibrium cutoff. Holding this benchmark fixed, the lemma shows that a higher ρ makes focality less responsive to current outcomes. Along histories with $\alpha^H > \alpha_0$, where wins dominate losses, a higher ρ lowers α^H by reducing the marginal effect of additional wins on future focality. Along histories with $\alpha^H < \alpha_0$, where losses dominate wins, a higher ρ raises α^H by reducing the adverse effect of additional losses. Hence, a higher ρ dampens both the dynamic benefit of success and the dynamic cost of failure. The overall effect on platform value therefore depends on how these two forces are aggregated across histories in equilibrium.

To compare these two forces, pair each history H_I with its complementary history H_E, which imply the same sequence of wins and losses with platform identities swapped. If both are non-absorbed, then

$$q + \beta(2\alpha^{H_I} - 1) + q + \beta(2\alpha^{H_E} - 1) = 2q + 2\beta\rho^t(2\alpha_0 - 1),$$

is increasing in ρ. Thus, the paired contribution of non-absorbed complementary histories rises with ρ.

Absorption reinforces this effect. At $\alpha_0 = \hat{\alpha}$, there is weakly more E-absorption than I-absorption. In particular, for any complementary pair (H_I, H_E), if H_I has been E-absorbed, then H_E must be I-absorbed as well, but the converse need not hold. The paired contribution of such complementary histories in platform I's value function is:

$$q + \beta(2\alpha^{H_E} - 1),$$

which increases in ρ. Therefore, taking into account the asymmetry in absorption, a higher ρ also raises platform I's value.

Combining together, after the increase in ρ, platform I still wins and receives a positive profit. Therefore, the threshold $\hat{\alpha}(q)$ decreases. The low-quality platform requires a lower initial focality level to win and the efficiency level falls.

Proposition 2 *As ρ increases, the equilibrium cutoff $\hat{\alpha}(q)$ decreases. The level of efficiency decreases.*

The intuition is that a higher ρ makes consumer focality more inertial. This reduces the effect of current outcomes on future coordination: additional wins raise future focality by less, but additional losses also lower future focality by less. At the equilibrium cutoff, the second effect is relatively more important for the low-quality platform. Greater persistence cushions the focality loss from unfavorable histories, and the asymmetry in absorption further shifts continuation values in its favor. As a result, the low-quality platform's value rises at the original cutoff.

The implication is that greater persistence weakens the role of quality in market selection. As focality becomes more persistent, an initial focality advantage of the low-quality platform is harder to overturn. Consequently, the low-quality platform wins for a larger set of initial conditions, and the level of efficiency falls.

Consumer Surplus We now study how consumer surplus changes with ρ in the equilibrium where platform I wins. Fix $\alpha_0 = \hat{\alpha}$, and consider an increase in ρ such that platform I still wins. Consumer surplus is given by

$$CS = \frac{q_E}{1-\delta} + \sum_{t=0}^{s^*} \delta^{t+1} V_E(\alpha^{I^t E}).$$

For $t \leq s^*$, we have $\alpha^{I^t E} < \hat{\alpha}$, and platform offers a subsidy equal to $V_E(\alpha^{I^t E})$.

As ρ increases, $\alpha^{I^t E}$ increases, which lowers $V_E(\alpha^{I^t E})$. In addition, at a given focality level, platform I's value increases while platform E's value decreases. Both effects reduce $V_E(\alpha^{I^t E})$, so platform E becomes less willing to compete. It then subsidizes less and for fewer periods. Hence, greater persistence reduces consumer surplus.

Proposition 3 *In an equilibrium where platform I wins, as ρ increases, platform E subsidizes less and for fewer periods. Consumer surplus decreases.*

Contrary to the usual view that greater persistence intensifies competition, here it weakens the high-quality platform's incentive to compete. The reason is that greater persistence makes the low-quality platform's initial focality advantage harder to overturn. This in turn reduces subsidies offered by the high-quality platform and lowers consumer surplus.

We have compared consumer surplus within the same equilibrium regime, where platform I wins both before and after the increase in ρ. Now suppose $\alpha_0 < \hat{\alpha}$, so that platform E wins initially. Since an increase in ρ lowers the threshold, α_0 may either remain below the threshold or cross above it. In the first case, platform E still wins, and consumer surplus increases because the low-quality platform has a stronger incentive to compete and subsidize. In the second case, the equilibrium winner switches from platform E to platform I. The change in consumer surplus then reflects both the change in the identity of the winning platform and the change in subsidies by the losing platform, so the net effect is hard to sign.

The benchmark with a common persistence parameter ρ isolates the effect of a uniform change in focality persistence across platforms. In that symmetric case, a higher ρ raises the overall persistence of focality for both platforms and lowers the equilibrium cutoff $\hat{\alpha}$. This captures the effect of the common level of persistence, holding fixed the symmetry of the updating process. A natural next step is therefore to relax this symmetry and allow the two platforms to differ in persistence. Doing so introduces an additional force: not only the overall level of persistence matters, but also which platform's focality is more persistent. The asymmetric case therefore allows us to separate a common-persistence effect from a relative-persistence effect.

4 Asymmetric focality updating

We now consider asymmetric focality dynamics, with $\rho_I \neq \rho_E$, so that one platform's focality is more persistent than the other's. Relative persistence across platforms therefore becomes an additional determinant of competitive advantage.

To isolate the effect of relative persistence, we first vary one persistence parameter while holding the other fixed. An increase in ρ_I, holding ρ_E fixed, means that platform I's focality declines less after losses. Hence, for any history H, platform I's focality level α^H is higher, which raises its discounted profit and lowers the equilibrium cutoff $\hat{\alpha}$.

Conversely, an increase in ρ_E, holding ρ_I fixed, makes platform E's focality more persistent, so platform I's focality rises less after wins. Thus, for any history H, platform I's focality level α^H is lower, which reduces its discounted profit and raises the equilibrium cutoff $\hat{\alpha}$.

When both ρ_I and ρ_E change, it is useful to distinguish between a change in relative persistence and a change in the overall level of persistence. If ρ_I increases while ρ_E decreases, both forces shift the cutoff in favor of platform I, so $\hat{\alpha}$ falls unambiguously. Likewise, if ρ_I decreases while ρ_E increases, both forces shift the cutoff in favor of platform E, so $\hat{\alpha}$ rises unambiguously. By contrast, if both persistence parameters increase, or both decrease, then the net effect on $\hat{\alpha}$ is generally ambiguous.

This ambiguity can be understood by decomposing the change into two effects. One is a common-persistence effect, which captures the effect of a common change in both persistence parameters, as in the symmetric benchmark. The other is a relative-persistence effect, which captures the change in persistence asymmetry across platforms. When both persistence parameters move in the same direction, the net effect on $\hat{\alpha}$ depends on the balance between these two effects.

The non-absorbing case illustrates this decomposition particularly clearly. When parameters satisfy $\rho_I \leq \hat{\alpha} \leq 1 - \rho_E$, there is no absorption on either side, and

$$V_I(\alpha_0) = \frac{q}{1-2\delta} + \frac{\beta}{1-\delta(\rho_I+\rho_E)}\left[(2\alpha_0 - 1) + \frac{\delta(\rho_I-\rho_E)}{1-2\delta}\right].$$

This expression shows that the overall level of persistence in market: $\rho_I + \rho_E$ and the relative persistence across platforms: $\rho_I - \rho_E$ enter separately. An increase in $\rho_I + \rho_E$ reflects a rise in the overall level of persistence and increases platform I's value, consistent with the common-persistence effect discussed in the symmetric benchmark. By contrast, an increase in $\rho_I - \rho_E$ strengthens platform I's relative persistence and thereby increases $V_I(\alpha_0)$. Thus, when both persistence parameters vary, the equilibrium effect reflects the interaction between these two components.

Proposition 4 *Suppose focality updates asymmetrically, with $\rho_I \neq \rho_E$. Then the equilibrium cutoff $\hat{\alpha}$ is decreasing in ρ_I and increasing in ρ_E:*

$$\frac{\partial \hat{\alpha}}{\partial \rho_I} < 0, \qquad \frac{\partial \hat{\alpha}}{\partial \rho_E} > 0.$$

Hence, an increase in ρ_I lowers the efficiency level, whereas an increase in ρ_E raises it.

The proposition shows that, under asymmetric focality updating, market selection depends not only on the initial focality level but also on which platform's focality is more persistent over time. Increasing ρ_I makes platform I's focality harder to erode after losses, whereas increasing ρ_E makes platform E's focality harder to erode after losses. The proposition therefore isolates the effect of relative persistence. More generally, the net effect of asymmetric persistence, (ρ_I, ρ_E), depends both on relative persistence across platforms and on the overall level of persistence in the market, that is, when there are uniform changes in the two platforms' persistence parameters. The equilibrium therefore depends both on the difference in persistence across platforms and on the overall level.

As discusses in the two-period model, the efficiency level need not be monotone in δ. An increase in δ can either raise or lower efficiency, and the direction depends on the degree of asymmetry in the focality process, $|\rho_I - \rho_E|$.

If there is no absorbing, i.e., for parameters such that $\rho_I < \hat{\alpha}(q) < 1 - \rho_E$, then

$$\hat{\alpha}(\delta) = \frac{1}{2} - \frac{q}{2\beta}\frac{1-(\rho_I+\rho_E)\delta}{1-2\delta} - \frac{\rho_I - \rho_E}{2}\frac{\delta}{1-2\delta}.$$

If $\rho_I \le \rho_E$, $\frac{\partial \hat{\alpha}}{\partial \delta} > 0$. If $\rho_I > \rho_E$, then

$$\frac{\partial \hat{\alpha}}{\partial \delta} > 0 \quad \Longleftrightarrow \quad q_E - q_I > \beta\frac{\rho_I - \rho_E}{2-\rho_I-\rho_E},$$

which is the same threshold for two-period model. Thus, when there is no asymmetry in absorption, the result reserves the structure of two-period model. When there is absorption, the result changes and we conclude in the following proposition.

Proposition 5 *Consider an exogenous asymmetric focality process (ρ_I, ρ_E) with $q < 0$.*

1. *for $\rho_I \le \rho_E$, the effect of δ in efficiency is always positive;*
2. *for $\rho_I > \rho_E$, the effect of δ in efficiency is positive if and only if:*

$$q_E - q_I > \phi(\rho_I - \rho_E),$$

where $\phi(\rho_I - \rho_E)$ is an increasing function of $\rho_I - \rho_E$.

When the high-quality platform's focality is weakly more persistent, the dynamics favor the high-quality platform. As δ increases, the low-quality platform would require a higher initial focality level to win, and the efficiency level increases. On the other hand, if the low-quality platform's focality is more persistent, the low-quality platform has an advantage in focality updating. The effect of δ in the efficiency level depends on the relative value of platforms' quality advantage, and the persistence asymmetry. When the quality gap is sufficiently large than the low-quality platform's persistence advantage, the effect of δ is positive.

5 Marketing investment

In the previous section, we treated platforms' focality dynamics, including the initial focality level and the focality updating process, as exogenous. We showed that, in equilibrium, one platform keeps winning while the other subsidizes. We examined how focality dynamics: the initial focality level and the focality updating determine platforms' incentives to compete and the equilibrium outcome. In this section, we endogenize the focality process by allowing platforms to invest in marketing.

Platforms can invest in marketing to improve focality along two dimensions:

- Level effect: a platform's marketing investment increases its own current focality level and decreases the rival's.
- Persistence effect: a platform's marketing investment increases the persistence of its own focality and decreases the persistence of the rival's, thereby affecting how focality updates over time.

Note that, under both effects, a platform's marketing investment shifts focality in its own favor and against its rival. As a result, the two platforms' marketing efforts work in opposite directions and may partly offset each other. The key difference between the two effects is that the level effect raises focality directly and is independent of the market outcome, whereas the persistence effect affects future focality only through the realized sequence of market outcomes.

For simplicity, we assume that both platforms make a one-time marketing investment simultaneously at the initial stage of the game. Let a_i denote platform I's marketing investment level, which determines α_0 and (ρ_I, ρ_E). After observing that, platforms set prices in each period and consumers make participation choice, as in the preceding discussion. This timing is natural both economically and analytically. From an economic perspective, platforms often concentrate marketing around product launch or commit to marketing budgets at an early stage. From an analytical perspective, our equilibrium is characterized by the interaction among the quality gap, the initial focality level, and the persistence parameters. Modeling marketing as an initial-stage choice therefore provides a parsimonious way to endogenize these key objects.

For simplicity, we assume that marketing has a constant marginal cost c.

We are interested in which platform has a stronger incentive to invest in marketing: the high-quality but initially less focal platform E, or the low-quality but initially more focal platform I. To isolate this question, we impose that, in the absence of marketing investment ($a_I = a_E = 0$), focality updating is symmetric with common persistence ρ, and platform I's initial focality level is exactly at the equilibrium cutoff, $\alpha_0 = \hat{\alpha}(q; \rho)$. In other words, we begin from a balance point at which platform E's quality advantage is exactly offset by platform I's focality advantage, and ask which platform has a stronger incentive to break the balance in its own favor through marketing investment.

The level effect and the persistence effect may shift platforms' marketing incentives in different directions. We study the two effects separately.

5.1 Level effect

We begin with the level effect, where marketing affects only the initial level of focality and does not alter the focality updating. In this case, focality evolves symmetrically across platforms with a common persistence parameter ρ.

Marketing shifts initial focality in a zero-sum way. A higher marketing investment by platform I raises its own initial focality and lowers that of platform E, while a higher investment by platform E has the opposite effect. To capture this idea in the simplest way, suppose that under marketing choices (a_I, a_E), platform I's initial focality in period zero is given by

$$\alpha_0 = \hat{\alpha} + a_I - a_E.$$

Under this specification, if both platforms choose the same marketing level, $a_I = a_E$, then the competing marketing campaigns cancel each other out and initial focality remains at $\hat{\alpha}$. The market remains at the balanced benchmark in which both platforms earn zero revenue in the pricing subgame. By contrast, if $a_I \neq a_E$, marketing shifts α_0 away from this balance point and thereby tilts the market in favor of one platform. We therefore study each platform's incentive to raise marketing investment relative to its rival.

A change in α_0 does not affect the threshold $\hat{\alpha}$, because the persistence parameters and the quality gap are held fixed. Intuitively, marketing that shifts initial focality is a direct substitute for subsidizing early participation. Instead of improving focality gradually through favorable early outcomes, a platform can shift focality immediately at the outset by changing α_0.

Platforms' payoffs from choosing (a_I, a_E) are

$$\begin{aligned} R_I(a_I, a_E) &= \max\{V_I(\hat{\alpha} + a_I - a_E), 0\} - a_I c, \\ R_E(a_I, a_E) &= \max\{V_E(\hat{\alpha} + a_I - a_E), 0\} - a_E c. \end{aligned}$$

When $a_I = a_E$, we have $V_I(\alpha_0) = V_E(\alpha_0) = 0$. Moreover, $V_I(\hat{\alpha} + a_I - a_E) > 0$ whenever $a_I > a_E$, while $V_E(\hat{\alpha} + a_I - a_E) > 0$ whenever $a_E > a_I$. Thus, a platform obtains a positive continuation value only if it sets a higher marketing level than its rival, implying corner best responses. Fix a_E. platform I either chooses $a_I = 0$ or chooses $a_I \geq a_E$; symmetrically, fixing a_I, platform E either chooses $a_E = 0$ or chooses $a_E \geq a_I$. Consequently, any pure strategy equilibrium must lie among the corner profiles: (i) $a_I = a_E = 0$; (ii) $a_I > 0, a_E = 0$; or (iii) $a_E > 0, a_I = 0$.

Thus, only the case in which one platform sets a higher marketing level than its rival is relevant. To study the incentive to invest more than the rival, define the deviation as the marketing lead over the rival.

If firm I sets a relatively higher marketing level, define $\ell_I \equiv a_I - a_E > 0$. Then

$$MR_I(\ell_I) = \frac{\partial}{\partial \ell_I} V_I(\hat{\alpha} + \ell_I) - c,$$

and it suffices to restrict attention to $\ell_I \in [0, 1 - \hat{\alpha}]$, since $V_I(\hat{\alpha} + \ell_I) = V_A(1)$ for all $\ell_I \geq 1 - \hat{\alpha}$.

If instead firm E sets a relatively higher level, define $\ell_E \equiv a_E - a_I > 0$. Then

$$MR_E(\ell_E) = \frac{\partial}{\partial \ell_E} V_E(\hat{\alpha} - \ell_E) - c,$$

and it suffices to restrict attention to $\ell_E \in [0, \hat{\alpha}]$, since $V_E(\hat{\alpha} - \ell_E) = V_E(0)$ for all $\ell_E \geq \hat{\alpha}$.

We then study how shifting α_0 changes the value functions. Recall that, starting from an initial focality level α_0, for any history H, I-absorption occurs when

$$\alpha^H > \frac{\hat{\alpha}}{\rho},$$

and E-absorption occurs when

$$\alpha^H < 1 - \frac{1 - \hat{\alpha}}{\rho},$$

where $\hat{\alpha}$ is fixed, and α^H increases in α_0. If a shift in the initial focality level has no effect on absorptions, for example, when $\rho < \hat{\alpha} < 1 - \rho$, so that no absorption occurs along any history, then the marginal return is a constant and

$$\frac{\partial V_I(\hat{\alpha} + \ell_I)}{\partial \ell_I} = \frac{\partial V_E(\hat{\alpha} - \ell_E)}{\partial \ell_E}.$$

In this case, the two platforms have symmetric marginal incentives in marketing investments.

By contrast, when absorption is reachable on at least one side, a shift in the initial focality level changes absorption, and marginal returns are not linear. If platform I sets a higher marketing level than platform E, then α^H is higher after each history. This shifts some histories toward I-absorption and away from E-absorption. As a result, platform I's marginal return to setting a higher marketing level relative to its rival is increasing. Symmetrically, if platform E increases its marketing lead relative to platform I, then α^H is lower after each history. This increases E-absorption and reduces I-absorption. Platform E's marginal return to setting a higher marketing level is increasing.

Lemma 2 *$MR_I(\ell_I)$ is weakly increasing on $[0, 1 - \hat{\alpha}]$, and $MR_E(\ell_E)$ is weakly increasing on $[0, \hat{\alpha}]$.*

Each platform's marginal return from setting a higher marketing level than its rival is weakly increasing in its marketing lead. Hence best responses are corner solutions, and it suffices to consider the two asymmetric candidate equilibria:

$$(a_I, a_E) = (1 - \hat{\alpha}, 0) \quad \text{and} \quad (a_I, a_E) = (0, \hat{\alpha}).$$

We next characterize the parameter conditions under which each candidate is an equilibrium.

Consider first $(a_I, a_E) = (1 - \hat{\alpha}, 0)$. Given $a_I = 1 - \hat{\alpha}$, platform E compares two actions: it can choose $a_E = 0$, in which case its payoff is $V_E(0)$, or it can out-market platform I by choosing $a_E = 1$, so that $\ell_E = \hat{\alpha}$, in which case its payoff is $V_E(0) - c\hat{\alpha}$. Thus platform E does not deviate if and only if

$$V_E(0) \leq c.$$

Next, given $a_E = 0$, platform I compares choosing $a_I = 0$, which yields payoff 0, with choosing $a_I = 1 - \hat{\alpha}$, which yields payoff $V_I(1) - c(1 - \hat{\alpha})$. Hence platform I prefers to choose $1 - \hat{\alpha}$ if and only if

$$\frac{V_I(1)}{1 - \hat{\alpha}} \geq c.$$

Therefore, $(1 - \hat{\alpha}, 0)$ is an equilibrium whenever

$$V_E(0) \leq c \leq \frac{V_I(1)}{1 - \hat{\alpha}}. \tag{4}$$

Now consider $(a_I, a_E) = (0, \hat{\alpha})$. By the same argument, this profile is an equilibrium whenever

$$V_I(1) \leq c \leq \frac{V_E(0)}{\hat{\alpha}}. \tag{5}$$

To compare the existence conditions for the two asymmetric corner equilibria, it is useful to compare the return to marketing for a platform when its rival sets marketing equal to zero. Accordingly, fix the rival's marketing level at zero, and suppose the two platforms choose positive marketing investments that induce the same initial focality level, that is,

$$\hat{\alpha} + \ell_I = 1 - \hat{\alpha} + \ell_E.$$

Then platform E earns a higher average return from marketing than platform I.

$$AR(\ell_I) < AR(\ell_E).$$

Intuitively, E's marketing shifts market outcomes more strongly: it reduces I-absorption by more and increases E-absorption by more than I's marketing does. It follows that

$$V_E(0) > V_I(1), \qquad \frac{V_E(0)}{\hat{\alpha}} > \frac{V_I(1)}{1 - \hat{\alpha}}.$$

Therefore, the set of parameter values for which only platform E invests in marketing is always nonempty. By contrast, the set of parameter values for which only platform I invests may be empty. In particular, the required condition may fail when $\hat{\alpha} > 1 - \rho + \rho^2$.

Hence, the parameter region supporting the E-only marketing equilibrium is larger. In particular, whenever an equilibrium in which only platform I invests exists, an equilibrium in which only platform E invests exists as well, whereas the converse does not hold.

Proposition 6 *Suppose firms choose marketing investments at the initial stage that shift the initial focality level. An equilibrium in which only firm I invests with* $a_I^* = 1 - \hat{\alpha}, a_E^* = 0$ *exists if:*

$$V_E(0) < c < \frac{V_I(1)}{1 - \hat{\alpha}}.$$

An equilibrium in which only firm E invests with $a_I^* = 0, a_E^* = \hat{\alpha}$ exists if:

$$V_I(1) < c < \frac{V_E(0)}{\hat{\alpha}}.$$

The set of parameters for which only firm E invests is a superset of the set for which only firm I invests.

When the cost of marketing is sufficiently high, so that $\frac{V_E(0)}{\hat{\alpha}} < c$, marketing is too expensive for either platform to find it profitable, and the unique equilibrium is that neither platform invests. At the other extreme, when the cost is sufficiently low, no pure-strategy equilibrium exists: each platform has an incentive to raise its marketing investment further in order to shift the initial focality level in its favor. The interesting case is therefore the intermediate range of marketing costs. In this region, marketing is valuable enough that at least one platform may wish to invest, but still costly enough that the platforms do not keep raising their marketing investments indefinitely.

In the intermediate region, the equilibrium in which only the high-quality platform invests is supported by a larger set of parameter values. The reason is that marketing is more effective for the high-quality platform E. For investments that increase a platform's initial focality to a certain level, it raises platform E's payoff by more than it raises platform I's payoff, because it shifts absorptions after any possible history more strongly toward E. Hence, marketing generates a larger increase in platform E's value function, which gives the high-quality platform a stronger incentive to invest. An implication is that, when only one platform advertises in equilibrium, it is more likely to be the high-quality platform than the initially more focal but lower-quality platform.

5.2 Persistence effect

We now turn to the persistence effect of marketing investment. In this case, marketing does not change the initial focality level, so α_0 remains fixed, but it affects the subsequent focality dynamics by endogenously changing the persistence parameters (ρ_I, ρ_E).

Specifically, marketing affects persistence as follows. Platform I's marketing makes platform I's focality more resilient when it loses, that is, it increases ρ_I, and makes platform E's focality less resilient when platform it wins, that is, it decreases ρ_E. Symmetrically, platform E's marketing increases ρ_E and decreases ρ_I. Accordingly, the persistence parameters depend on (a_I, a_E) only through the difference $a_I - a_E$. For simplicity, we adopt the following parametric specification:

$$\rho_I(a_I, a_E) = \rho + a_I - a_E, \qquad \rho_E(a_I, a_E) = \rho + a_E - a_I.$$

As defined previously, let $\ell_i = a_i - a_j$ whenever $a_i > a_j$. We then write

$$V_i(\ell_i) = V_i(\alpha_0 \mid \rho_i = \rho + \ell_i,\ \rho_j = \rho - \ell_i),$$

where $V_I(\ell_i > 0)$ if and only if $\ell_i > 0$. Platforms' marginal incentives to increase marketing relative to the rival are

$$
\begin{aligned}
MR_I &= \frac{\partial V_I(\ell_I)}{\partial \ell_I} - c, \\
MR_E &= \frac{\partial V_E(\ell_E)}{\partial \ell_E} - c.
\end{aligned}
$$

It is sufficient to restrict attention to the region in which $\ell_i < \bar{a} = \max\{\rho, 1-\rho\}$.

As in the previous case, any equilibrium must be one of the corner profiles: (i) $a_I = a_E = 0$; (ii) $a_I > 0,\ a_E = 0$; or (iii) $a_E > 0,\ a_I = 0$.

We then study how changes in (ρ_I, ρ_E) affect the value functions. When ρ_I increases relative to ρ_E, α^H rises for every history H, which directly raises platform I's value function. At the same time, the change in persistence may alter the absorptions, and thus which histories contribute to the value function. A similar argument applies when ρ_E increases relative to ρ_I.

First, consider the case when change in persistence does not affect absorptions. That is, when $\rho < \hat{\alpha}(q;\rho) < 1-\rho$, and ℓ_i relatively small such that $\rho_j < \hat{\alpha}(q;\rho_I,\rho_E) < 1-\rho_i$. Then the change in the relative persistence only change platforms' value functions by the level increase or decrease in α_H. A marginal increase in platform I's relative marketing has the same effect on its continuation value as a marginal increase in platform E's relative marketing has on its continuation value. Hence,

$$
\frac{\partial V_I(\ell_I)}{\partial \ell_I} = \frac{\partial V_E(\ell_E)}{\partial \ell_E},
$$

and therefore $MR_I = MR_E$ locally.

When absorptions are reachable on at least one side, an increase in the relative marketing level would have a higher marginal return. That is, the marginal returns of increasing the marketing level relative to the rival are increasing in the gap for both platforms. Moreover, the marginal returns are not symmetric across two platforms. Note that, E-absorption is reachable when

$$
\ell_E > 1 - \hat{\alpha} - \rho,
$$

while I-absorption is reachable when

$$
\ell_I > \hat{\alpha} - \rho.
$$

Since $\hat{\alpha} > \frac{1}{2}$, we have

$$
1 - \hat{\alpha} - \rho < \hat{\alpha} - \rho.
$$

Starting from the no-absorption region, a smaller marketing lead is sufficient for platform E to make E-absorption reachable than for platform I to make I-absorption reachable. As discussed above, this reflects the fact that the absorption structure favors the high-quality platform E: E-absorption is easier to reach than I-absorption. Consequently, for the same level of marketing lead, platform E's marketing increases E-absorption and decreases I-absorption by more than platform I's marketing increases I-absorption and decreases E-absorption. Therefore, the marginal return to setting a higher marketing level is larger for platform E than for platform I.

Lemma 3 *$MR_I(\ell_I)$ and $MR_E(\ell_E)$ are weakly increasing, and for $\ell_I = \ell_E$,*

$$MR_I(\ell_I) \leq MR_E(\ell_E).$$

We then compare the two equilibrium candidates: $(a_I, a_E) = (\bar{a}, 0)$ and $(a_I, a_E) = (0, \bar{a})$. An equilibrium in which only platform I invests exists when

$$\frac{V_E(\bar{a})}{2\bar{a}} < c < \frac{V_I(\bar{a})}{\bar{a}},$$

whereas an equilibrium in which only platform E invests exists when

$$\frac{V_I(\bar{a})}{2\bar{a}} < c < \frac{V_E(\bar{a})}{\bar{a}}.$$

Since $V_E(\ell) > V_I(\ell)$, the second interval is larger. Therefore, the equilibrium in which only platform E invests exists for a larger set of parameter values.

Proposition 7 *Suppose firms choose marketing investments at the initial stage that shift persistence of platforms. The equilibrium in which only firm I invests with $a_I^* = \bar{a}, a_E^* = 0$ exists if*

$$\frac{V_E(\bar{a})}{2\bar{a}} < c < \frac{V_I(\bar{a})}{\bar{a}}.$$

The equilibrium in which only firm E invests with $a_I^* = 0, a_E^* = \bar{a}$ exists if

$$\frac{V_I(\bar{a})}{2\bar{a}} < c < \frac{V_E(\bar{a})}{\bar{a}}.$$

The set of parameters for which only firm E invests is a superset of the set for which only firm I invests.

Similar to the level-effect case, when the marketing cost is sufficiently high, neither platform invests in equilibrium. When the cost is sufficiently low, no pure-strategy equilibrium exists. For intermediate marketing costs, the equilibrium in which only the high-quality platform invests is supported by a larger set of parameter values. This is because the high-quality platform gains more from improving its focality updating process, and therefore has a stronger incentive to invest in marketing than the low-quality platform.

In sum, the level effect and the persistence effect both shift marketing incentives in the same direction. Whether marketing improves the initial focality level or the persistence of focality updating, the high-quality platform has the stronger incentive to invest.

6 Conclusion

We develop a tractable model of focality dynamics in platform competition. Consumer beliefs evolve through exponentially weighted updating, so current outcomes affect future focality and thereby shape future competition. This framework yields sharp comparative statics in an infinite-horizon setting.

The comparative statics of focality dynamics can be decomposed into two forces. First, there is a common shift in the persistence of both platforms' focality. An increase in this common persistence makes early expectation advantages more difficult to reverse, weakens the high-quality platform's incentive to compete aggressively, and lowers consumer surplus. Second, there is a change in relative persistence across platforms. When one platform's focality becomes more persistent relative to its rival's, competition shifts in its favor. The overall effect of changes in focality persistence is therefore governed by the interaction of these two forces.

The effect of dynamics on market efficiency reflects both platforms' intertemporal discounting and focality dynamics. This overall effect is positive when the quality advantage of the high-quality platform is relatively large than its disadvantage in focality updating.

Finally, we study separately the level effect and the persistence effect of platforms' marketing investment choices. Both effects shift platforms' incentives to invest in the same direction. It may also be possible to compare the relative strength of these two effects. More broadly, the paper highlights that focality should be treated not only as a static force, but also as a dynamic state variable shaped by market history. This provides a tractable foundation for future work on endogenous focality and on platform strategies designed to influence how consumers coordinate over time.

7 Appendix

This appendix presents the proofs of lemmas and propositions.

Proof for proposition 1

Fix $q \equiv q_I - q_E$. For each finite horizon $T \geq 0$, let $V_I^T(\alpha)$ and $V_E^T(\alpha)$ denote the equilibrium continuation payoffs when T periods remain and the current focality state is α. Set terminal values

$$V_I^0(\alpha) = V_E^0(\alpha) = 0 \qquad \text{for all } \alpha \in [0, 1].$$

We first show that, for every finite horizon T, the game admits a unique subgame perfect equilibrium.

Fix a subgame with $T \geq 1$ periods remaining and current state α. Given continuation values (V_I^{T-1}, V_E^{T-1}), platform E's lowest price consistent with winning the current period is

$$p_E = -\delta V_E^{T-1}(f_E(\alpha)),$$

since any lower price would make it strictly prefer to lose today and obtain its continuation payoff instead. Hence, if platform I wins, it can do so by setting

$$p_I = q + \beta(2\alpha - 1) - \delta V_E^{T-1}(f_E(\alpha)),$$

which yields total payoff

$$q + \beta(2\alpha - 1) + \delta V_I^{T-1}(f_I(\alpha); q) - \delta V_E^{T-1}(f_E(\alpha)).$$

Define

$$\Delta_T(\alpha) := q + \beta(2\alpha - 1) + \delta V_I^{T-1}(f_I(\alpha)) - \delta V_E^{T-1}(f_E(\alpha)).$$

Symmetrically, if platform E wins, its total payoff is $-\Delta_T(\alpha)$. Therefore, given the continuation values, the current-period pricing subgame has the following equilibrium outcome: platform I wins if $\Delta_T(\alpha) > 0$, platform E wins if $\Delta_T(\alpha) < 0$, and if $\Delta_T(\alpha) = 0$, both firms are indifferent. We break ties in favor of platform I. This affects only the knife-edge case $\Delta_T(\alpha; q) = 0$ and guarantees uniqueness. Hence the equilibrium continuation payoffs satisfy

$$V_I^T(\alpha) = \max\{\Delta_T(\alpha), 0\}, \qquad V_E^T(\alpha) = \max\{-\Delta_T(\alpha), 0\}. \tag{1}$$

Equation (1) uniquely determines (V_I^T, V_E^T) from (V_I^{T-1}, V_E^{T-1}). Since (V_I^0, V_E^0) is given, backward induction implies that, for every finite horizon T, the game admits a unique subgame perfect equilibrium.

Next we characterize each finite-horizon equilibrium by a cutoff. By assumption, for every T and q, the function $\Delta_T(\alpha; q)$ is continuous and strictly increasing in α. Hence there exists a unique threshold $\hat{\alpha}_T(q) \in [0, 1]$ such that

$$\Delta_T(\hat{\alpha}_T(q)) = 0,$$

and

$$\Delta_T(\alpha) \geq 0 \iff \alpha \geq \hat{\alpha}_T(q).$$

Therefore, in the unique T-period equilibrium, platform I wins if and only if

$$\alpha \geq \hat{\alpha}_T(q).$$

It remains to show that the thresholds converge as $T \to \infty$.

Suppose

$$\Delta_T(\alpha) \leq \Delta_{T-1}(\alpha) \qquad \text{for all } \alpha \in [0, 1].$$

Since each Δ_T is strictly increasing in α, this implies

$$\{\alpha : \Delta_T(\alpha) \geq 0\} \subseteq \{\alpha : \Delta_{T-1}(\alpha) \geq 0\},$$

and hence

$$\hat{\alpha}_T(q) \geq \hat{\alpha}_{T-1}(q).$$

Moreover,

$$\begin{aligned}
\Delta_{T+1}(\alpha) - \Delta_T(\alpha) &= \delta\Big(V_I^T(f_I(\alpha)) - V_I^{T-1}(f_I(\alpha))\Big) - \delta\Big(V_E^T(f_E(\alpha)) - V_E^{T-1}(f_E(\alpha))\Big) \\
&= \delta\Big(\max\{\Delta_T(f_I(\alpha)), 0\} - \max\{\Delta_{T-1}(f_I(\alpha)), 0\}\Big) \\
&\quad - \delta\Big(\max\{-\Delta_T(f_E(\alpha)), 0\} - \max\{-\Delta_{T-1}(f_E(\alpha)), 0\}\Big).
\end{aligned}$$

Because $\Delta_T \leq \Delta_{T-1}$ pointwise, the first bracket is weakly negative and the second bracket is weakly positive. Therefore

$$\Delta_{T+1}(\alpha) - \Delta_T(\alpha) \leq 0 \qquad \text{for all } \alpha.$$

So $\Delta_{T+1} \leq \Delta_T$ pointwise, which implies

$$\hat{\alpha}_{T+1}(q) \geq \hat{\alpha}_T(q).$$

So, it depends on the base step, $\hat{\alpha}_1(q)$ and $\hat{\alpha}_2(q)$. With

$$\hat{\alpha}_1(q) = \frac{1}{2} - \frac{q}{2\beta}, \quad \hat{\alpha}_2(q) = -\frac{(1+2\delta)(q-\beta)}{2\beta} - \delta(f_I(\hat{\alpha}_1(q)) + f_E(\hat{\alpha}_1(q)) - 1).$$

For $q < 0$, we have $\hat{\alpha}_2(q) > \hat{\alpha}_1(q)$. Thus, by induction, we have $\hat{\alpha}_T(q) > \hat{\alpha}_{T-1}(q)$.

Thus $\{\hat{\alpha}_T(q)\}_{T\geq 1}$ is a monotone sequence. Because $\hat{\alpha}_T(q) \in [0, 1]$ for every T, the sequence is bounded. Hence $\hat{\alpha}_T(q)$ converges. Denote its limit by

$$\hat{\alpha}(q) := \lim_{T\to\infty} \hat{\alpha}_T(q) \in [0, 1].$$

We have shown that the finite-horizon equilibrium cutoff $\hat{\alpha}_T(q)$ converges to some $\hat{\alpha}(q) \in [0,1]$. Consider the cutoff rule

$$\alpha \geq \hat{\alpha}(q) \quad \Longleftrightarrow \quad \text{platform } I \text{ wins},$$

with ties broken in favor of I. Let V_I, V_E denote the continuation payoffs generated by iterating this rule forward. Then these payoffs satisfy

$$V_I(\alpha) = \max\{\Delta(\alpha), 0\}, \qquad V_E(\alpha) = \max\{-\Delta(\alpha), 0\},$$

where

$$\Delta(\alpha) = q + \beta(2\alpha - 1) + \delta V_I(f_I(\alpha)) - \delta V_E(f_E(\alpha)).$$

Moreover, by construction of the cutoff,

$$\Delta(\alpha) \geq 0 \iff \alpha \geq \hat{\alpha}(q).$$

Hence the cutoff rule $\hat{\alpha}(q)$ is an equilibrium of the infinite-horizon game.

Finally, for every $\alpha \neq \hat{\alpha}(q)$, the finite-horizon equilibrium outcome is eventually constant and coincides with the cutoff rule above, since $\hat{\alpha}_T(q) \to \hat{\alpha}(q)$. Therefore the infinite-horizon equilibrium is the limit of the finite-horizon equilibria. □

Proof for corollary 1

We first show for $\hat{\alpha}_T(q, \delta)$. Consider finite T.

$$\hat{\alpha}_T = \tfrac{1}{2} - \frac{q}{\beta} \frac{\sum_{t=0}^{T-1} n_t \delta^t}{\sum_{t=0}^{T-1} n_t \rho^t \delta^t} + \frac{K_T}{2M_T}$$

For finite T, let $\mathcal{S}_t$ denote the set of surviving (non-pruned) histories of length t, and define

$$n_t := |\mathcal{S}_t|.$$

Thus n_t is just the number of histories that still remain at date t.

Under the symmetric transition rule

$$f_I(\alpha) = 1 - \rho + \rho\alpha, \qquad f_E(\alpha) = \rho\alpha,$$

for every surviving history H of length t we can write

$$2\alpha^H - 1 = k_H + \rho^t(2\alpha_0 - 1),$$

where k_H is a history-specific constant that does not depend on α_0.

Hence the pruned value difference can be written as

$$\Delta_T(\alpha_0) = \sum_{t=0}^{T-1} \sum_{H\in\mathcal{S}_t} \delta^t[q+\beta(2\alpha^H-1)]$$

$$= q\sum_{t=0}^{T-1} n_t\delta^t + \beta\sum_{t=0}^{T-1}\delta^t \sum_{H\in\mathcal{S}_t} k_H + \beta(2\alpha_0-1)\sum_{t=0}^{T-1} n_t\rho^t\delta^t.$$

Now define

$$M_T := \sum_{t=0}^{T-1} n_t\rho^t\delta^t, \qquad K_T := -\sum_{t=0}^{T-1}\delta^t\sum_{H\in\mathcal{S}_t} k_H.$$

So M_T is the total coefficient on $2\alpha_0 - 1$, while K_T collects all path-dependent constant terms.

Then

$$\Delta_T(\alpha_0) = q\sum_{t=0}^{T-1} n_t\delta^t - \beta K_T + \beta(2\alpha_0-1)M_T.$$

Thus,

$$\hat{\alpha}_T = \frac{1}{2} - \frac{q}{2\beta}\frac{\sum_{t=0}^{T-1} n_t\delta^t}{\sum_{t=0}^{T-1} n_t\rho^t\delta^t} + \frac{K_T}{2M_T}.$$

It can be seen that

$$\frac{\partial^2\hat{\alpha}_T}{\partial q\partial\delta} = \frac{\partial}{\partial\delta}\left(-\frac{\sum_{t=0}^{T-1} n_t\delta^t}{2\sum_{t=0}^{T-1} n_t\rho^t\delta^t}\right) < 0.$$

Take $T \to \infty$, we have $\frac{\partial^2\hat{\alpha}}{\partial q\partial\delta} < 0$. Since $\hat{\alpha}$ is twice differentiable, we have $\frac{\partial^2\hat{\alpha}}{\partial\delta\partial q} < 0$. Since at $q \to 0^+$, $\frac{\partial\hat{\alpha}(q)}{\partial\delta} > 0$, we have For all q, $\hat{\alpha}(q)$ increasing in δ. □

Proof for lemma 1

For histories that consists of the same outcome in every period, i.e., I^t or E^t, they are always non-absorbing, and it is easy to show that α^{I^t} decreases in ρ, while α^{E^t} increases in ρ.

Besides these, we focus on histories that are not one firm consistent winning/losing, with at least one alternation. For each of such history, we can decompose it into blocks. Define the two families of blocks

$$\mathcal{B}^{EI} := \{\, E^s I : \; s \in \mathbb{Z}_{\geq 0}\,\}, \qquad \mathcal{B}^{IE} := \{\, I^t E : \; t \in \mathbb{Z}_{\geq 0}\,\}.$$

Every history H admits a unique representation as a finite concatenation of blocks with alternating types from $\mathcal{B}^{EI}$ and $\mathcal{B}^{IE}$. Furthermore, denote H_I as the history ending with I winning in the last period, and similar for H_E. Then, we consider histories that are of the following form

$$H_E = (E^{s_1}II^{t_1}E)\cdots(E^{s_n}II^{t_n}E), \qquad H_I = (I^{s_1}EE^{t_1}I)\cdots(I^{s_n}EE^{t_n}I),$$

for $s_i, t_i \in \mathbb{Z}_{\geq 0}$. Moreover, for any H_I and H_E of the above form, we restrict that for

$$\alpha^{H_I} > \alpha_0, \quad \alpha^{H_E} < \alpha_0.$$

This ensures that neither history has reached absorption. Equivalently, this requires: $s_i < s_i^*$, and $t < t_i^*$.

First, we show that α^{H_E} is increasing in ρ.

1. We begin with the case $n = 1$. We show that

$$\alpha^{E^s II^t E}$$

is increasing in ρ for all $0 \leq s \leq s_1^*$ and $0 \leq t \leq t_1^*$.

Under EWMA,

$$\alpha^{E^s II^t E} = \rho - \rho^{t+2} + \rho^{t+s+2}\alpha_0 =: f(\rho),$$

so

$$f'(\rho) = 1 - (t+2)\rho^{t+1} + (t+s+2)\alpha_0\rho^{t+s+1}.$$

We will show that $f'(\rho) > 0$.

Step 1. A lower bound for $f'(\rho)$. Since $\alpha^{E^s II^t E} < \alpha_0$, we have

$$\rho - \rho^{t+2} + \rho^{t+s+2}\alpha_0 < \alpha_0.$$

Rearranging gives

$$\rho(1 - \rho^{t+1}) < \alpha_0(1 - \rho^{t+s+2}),$$

or equivalently,

$$\alpha_0 > \underline{\alpha}(\rho) := \frac{\rho(1 - \rho^{t+1})}{1 - \rho^{t+s+2}}.$$

Because $f'(\rho)$ is increasing in α_0, it follows that

$$f'(\rho) > 1 - (t+2)\rho^{t+1} + (t+s+2)\underline{\alpha}(\rho)\rho^{t+s+1}.$$

Substituting for $\underline{\alpha}(\rho)$ and simplifying,

$$f'(\rho) > \frac{G(\rho)}{1 - \rho^{t+s+2}},$$

where

$$\begin{aligned} G(\rho) &:= (1 - \rho^{t+1})(1 + s\rho^{t+s+2}) - (t+1)\rho^{t+1}(1 - \rho^{s+1}) \\ &= 1 - (t+2)\rho^{t+1} + \rho^{t+s+2}((t+s+1) - s\rho^{t+1}). \end{aligned}$$

Since $1-\rho^{t+s+2} > 0$, it is enough to prove that $G(\rho) > 0$.

Step 2. A restriction on ρ. We also know that $\alpha^{E^sI} > \alpha_0$. Under EWMA,

$$\alpha^{E^sI} = (1-\rho) + \rho^{s+1}\alpha_0,$$

so $\alpha^{E^sI} > \alpha_0$ is equivalent to

$$1-\rho > \alpha_0(1-\rho^{s+1}),$$

that is,

$$\alpha_0 < \frac{1-\rho}{1-\rho^{s+1}}.$$

Together with $\alpha_0 > \frac{1}{2}$, this implies

$$\frac{1}{2} < \frac{1-\rho}{1-\rho^{s+1}},$$

or

$$1+\rho+\cdots+\rho^s < 2. \tag{2}$$

Equivalently,

$$\frac{1-\rho^{s+1}}{1-\rho} < 2 \quad \Longleftrightarrow \quad \rho^{s+1} > 2\rho - 1.$$

Step 3. Proof that $G(\rho) > 0$. We consider three cases.

Case 1: $s = 0$.

In this case,

$$G(\rho) = (1-\rho^{t+1}) - (t+1)\rho^{t+1}(1-\rho).$$

Hence

$$G(\rho) = (1-\rho)\left(\frac{1-\rho^{t+1}}{1-\rho} - (t+1)\rho^{t+1}\right).$$

Now

$$\frac{1-\rho^{t+1}}{1-\rho} = 1+\rho+\cdots+\rho^t > (t+1)\rho^{t+1},$$

since each term ρ^k with $0 \le k \le t$ is strictly larger than ρ^{t+1}. Therefore $G(\rho) > 0$.

Case 2: $s = 1$.

Now

$$G(\rho) = (1-\rho^{t+1})(1+\rho^{t+3}) - (t+1)\rho^{t+1}(1-\rho^2),$$

which simplifies to

$$G(\rho) = 1 - (t+2)\rho^{t+1} + (t+2)\rho^{t+3} - \rho^{2t+4}.$$

Write $F_t(\rho) := G(\rho)$. Then

$$F_{t+1}(\rho) - F_t(\rho) = \rho^{t+1}(1-\rho^2)((t+2) - (t+3)\rho + \rho^{t+3}).$$

It remains to show that the bracketed term is positive. Define

$$\phi(\rho) := (t+2) - (t+3)\rho + \rho^{t+3}.$$

Since $\phi(1) = 0$ and

$$\phi'(\rho) = -(t+3) + (t+3)\rho^{t+2} = (t+3)(\rho^{t+2} - 1) < 0 \qquad \text{for } \rho \in (0,1),$$

we have $\phi(\rho) > 0$ on $(0,1)$. Hence $F_{t+1}(\rho) > F_t(\rho)$, so

$$F_t(\rho) \geq F_0(\rho).$$

Finally,

$$F_0(\rho) = 1 - 2\rho + \rho^3(2-\rho) = (1-\rho)^3(1+\rho) > 0.$$

Therefore $G(\rho) = F_t(\rho) > 0$ for all $t \geq 0$.

Case 3: $s \geq 2$.

From (2),

$$1 + \rho + \rho^2 \leq 1 + \rho + \cdots + \rho^s < 2,$$

so

$$\rho^2 + \rho - 1 < 0.$$

Let

$$\hat{\rho} := \frac{\sqrt{5}-1}{2},$$

the positive root of $\rho^2 + \rho - 1 = 0$. Then necessarily $\rho < \hat{\rho}$.

We first consider $t = 0$. In that case,

$$G(\rho) = 1 - 2\rho + \rho^{s+2}(s + 1 - s\rho).$$

If $\rho < \frac{1}{2}$, then $1 - 2\rho > 0$, so clearly $G(\rho) > 0$.

Now suppose $\rho \geq \frac{1}{2}$. Since $s \geq 2$, we have $\frac{1}{2} \geq \frac{1}{s}$, hence

$$\rho(s + 1 - s\rho) \geq 1.$$

Using also $\rho^{s+1} > 2\rho - 1$, we obtain

$$G(\rho) = 1 - 2\rho + \rho^{s+1}\rho(s + 1 - s\rho) > 1 - 2\rho + \rho^{s+1} > 1 - 2\rho + (2\rho - 1) = 0.$$

Next consider $t \geq 1$. From the expression for $G(\rho)$,

$$(t + s + 1) - s\rho^{t+1} = (t+1) + s(1 - \rho^{t+1}) \geq t + 1,$$

so

$$G(\rho) \geq 1 - (t+2)\rho^{t+1} + (t+1)\rho^{t+s+2}. \tag{3}$$

Using $\rho^{s+1} > 2\rho - 1$, we have

$$\rho^{t+s+2} = \rho^{t+1}\rho^{s+1} > \rho^{t+1}(2\rho - 1).$$

Substituting this into (3) yields

$$G(\rho) > 1 - \rho^{t+1}(2t + 3 - 2(t+1)\rho). \tag{4}$$

Define

$$p_t(\rho) := \rho^{t+1}(2t + 3 - 2(t+1)\rho).$$

As shown earlier, for each fixed $\rho \in (0, \hat{\rho}]$, the sequence $p_t(\rho)$ is weakly decreasing in $t \geq 1$. Hence

$$p_t(\rho) \leq p_1(\rho).$$

Moreover,

$$p_1(\rho) = \rho^2(5 - 4\rho)$$

is increasing on $(0, 5/6)$, and $\hat{\rho} < 5/6$. Therefore

$$p_t(\rho) \leq p_1(\rho) \leq p_1(\hat{\rho}).$$

Using $\hat{\rho}^2 = 1 - \hat{\rho}$,

$$p_1(\hat{\rho}) = (1 - \hat{\rho})(5 - 4\hat{\rho}) = \frac{31 - 13\sqrt{5}}{2} < 1.$$

So $p_t(\rho) < 1$ for all $t \geq 1$ and $\rho \in (0, \hat{\rho}]$. In particular, at our $\rho < \hat{\rho}$, (4) implies $G(\rho) > 0$.

In all cases, $G(\rho) > 0$. Therefore $f'(\rho) > 0$, and hence $\alpha^{E^s II^t E}$ is increasing in ρ.

2. For $n > 1$: Define recursively

$$a_0 := \alpha_0, \qquad a_i := \alpha^{(E^{s_1} II^{t_1} E)\cdots(E^{s_i} II^{t_i} E)}, \qquad i = 1, \ldots, n.$$

Then $a_n = \alpha^{H_E}$. Under EWMA, each block updates focality according to

$$a_i = \rho - \rho^{t_i+2} + \rho^{t_i+s_i+2} a_{i-1}.$$

Differentiating with respect to ρ,

$$\frac{da_i}{d\rho} = 1 - (t_i + 2)\rho^{t_i+1} + (t_i + s_i + 2)\rho^{t_i+s_i+1} a_{i-1} + \rho^{t_i+s_i+2} \frac{da_{i-1}}{d\rho}.$$

Rewrite this as

$$\frac{da_i}{d\rho} = \left[1 - (t_i + 2)\rho^{t_i+1} + (t_i + s_i + 2)\rho^{t_i+s_i+1}a_0\right] + (t_i + s_i + 2)\rho^{t_i+s_i+1}(a_{i-1} - a_0) + \rho^{t_i+s_i+2}\frac{da_{i-1}}{d\rho}.$$

By the $n = 1$ case, applied to the block $E^{s_i}II^{t_i}E$, the assumptions

$$\alpha^{E^{s_i}I} > a_0, \qquad \alpha^{E^{s_i}II^{t_i}E} < a_0$$

imply

$$1 - (t_i + 2)\rho^{t_i+1} + (t_i + s_i + 2)\rho^{t_i+s_i+1}a_0 > 0.$$

Moreover, by construction $a_{i-1} > a_0$, since after each partial history ending in I, focality remains above the initial level. Hence

$$(t_i + s_i + 2)\rho^{t_i+s_i+1}(a_{i-1} - a_0) \geq 0.$$

We now prove by induction that $da_i/d\rho > 0$ for all i.

For $i = 1$, this is exactly the $n = 1$ result. Suppose $da_{i-1}/d\rho > 0$. Then every term on the right-hand side above is nonnegative, and the first term is strictly positive. Therefore

$$\frac{da_i}{d\rho} > 0.$$

By induction, $da_n/d\rho > 0$. Since $a_n = \alpha^{H_E}$, it follows that α^{H_E} is increasing in ρ.

Using symmetric method, we can show that α^{H_I} is decreasing in ρ. □

Proof for Proposition 2

Recall that:

$$V_I(\alpha_0) = \sum_{H_I \in \mathcal{H}^N} \delta^{|H_I|}(q + \beta(2\alpha^{H_I} - 1)) + \sum_{H_E \in \mathcal{H}^N} \delta^{|H_E|}(q + \beta(2\alpha^{H_E} - 1)),$$

Consider a pair of complementary histories (H_I, H_E). If both are non-absorbed, then

$$q + \beta(2\alpha^{H_I} - 1) + q + \beta(2\alpha^{H_E} - 1) = 2q + 2\beta\rho^t(2\alpha_0 - 1),$$

is increasing in ρ.

As shown previously, at $\alpha_0 = \hat{\alpha}$, there is more E-absorption than I-absorption. For a complementary pair (H_I, H_E), if H_I has been E-absorbed, then H_E must be I-absorbed as well, but the converse need not hold. The paired contribution of such complementary histories in platform I's value function is:

$$q + \beta(2\alpha^{H_E} - 1),$$

which increases in ρ. Thus, $V_I(\alpha_0)$ increases in ρ. It follows immediate that $\hat{\alpha}(q)$ decreases in ρ. □

Proof for Proposition 3

Take $\alpha_0 = \hat{\alpha}$, we show that

$$CS = \frac{q_E}{1-\delta} + \sum_{t=0}^{s^*} \delta^{t+1} V_E(\alpha^{I^t E})$$

decreases in ρ.

Since we have shown that

$$V_I(\alpha_0) = \sum_{t=0}^{\infty} (q_I - q_E + \beta(2\alpha^{I^t} - 1)) - \sum_{t=0}^{s^*} \delta^{t+1} V_E(\alpha^{I^t E})$$

increases in ρ, and α^{I^t} decreases in ρ, it follows immediate that CS decreases. □

Proof for Proposition 4

Since

$$\frac{\partial \alpha^H}{\partial \rho_I} > 0, \quad \frac{\partial \alpha^H}{\partial \rho_E} < 0,$$

it follows immediately that $V_I(\alpha_0)$ increases in ρ_I, and decreases in ρ_E, and thus

$$\frac{\partial \hat{\alpha}}{\partial \rho_I} < 0, \qquad \frac{\partial \hat{\alpha}}{\partial \rho_E} > 0.$$

□

Proof for Proposition 5

We first show for finite T. Fix a horizon T, and consider a parameter region on which the pruning pattern is fixed. Let $\mathcal{S}_T$ denote the set of surviving histories H with $|H| \le T-1$. For each $H \in \mathcal{S}_T$, write

$$2\alpha^H - 1 = A_H + B_H(2\alpha - 1),$$

where in the asymmetric case

$$B_H = \rho_E^{n_I(H)} \rho_I^{n_E(H)},$$

and $n_I(H)$, $n_E(H)$ are the numbers of I- and E-moves in H.

Define

$$N_T(\delta) := \sum_{H \in \mathcal{S}_T} \delta^{|H|}, \qquad M_T(\delta) := \sum_{H \in \mathcal{S}_T} \delta^{|H|} B_H, \qquad K_T(\delta) := \sum_{H \in \mathcal{S}_T} \delta^{|H|} A_H.$$

Then the cutoff $\hat{\alpha}_T(\delta)$ solving $\Delta_T(\hat{\alpha}_T; \delta) = 0$ is

$$\hat{\alpha}_T(\delta) = \frac{1}{2} - \frac{q}{2\beta} \frac{N_T(\delta)}{M_T(\delta)} - \frac{1}{2} \frac{K_T(\delta)}{M_T(\delta)}.$$

Moreover, if we define

$$\omega_H(\delta) := \frac{\delta^{|H|} B_H}{M_T(\delta)}, \qquad \lambda_H := \frac{A_H}{B_H},$$

then

$$2\delta \, \frac{\partial \hat{\alpha}_T}{\partial \delta} = \frac{q_E - q_I}{\beta} \, \mathrm{Cov}_\omega\Big(|H|, \frac{1}{B_H}\Big) - \mathrm{Cov}_\omega(|H|, \lambda_H).$$

Here $\mathrm{Cov}_\omega(x_H, y_H)$ denotes

$$\mathrm{Cov}_\omega(x_H, y_H) = \sum_{H \in \mathcal{S}_T} \omega_H x_H y_H - \left(\sum_{H \in \mathcal{S}_T} \omega_H x_H \right) \left(\sum_{H \in \mathcal{S}_T} \omega_H y_H \right).$$

We aim to show that:

1. if

$$\mathrm{Cov}_\omega\Big(|H|, \frac{1}{B_H}\Big) \geq 0 \quad \text{and} \quad \mathrm{Cov}_\omega(|H|, \lambda_H) \leq 0,$$

   then $\hat{\alpha}_T$ is increasing in δ;

2. if

$$\mathrm{Cov}_\omega\Big(|H|, \frac{1}{B_H}\Big) > 0,$$

   then

$$\frac{\partial \hat{\alpha}_T}{\partial \delta} > 0 \quad \Longleftrightarrow \quad q_E - q_I > \phi_T(\delta),$$

   where

$$\phi_T(\delta) := \beta \, \frac{\mathrm{Cov}_\omega(|H|, \lambda_H)}{\mathrm{Cov}_\omega\Big(|H|, \frac{1}{B_H}\Big)}.$$

Since

$$\Delta_T(\alpha; \delta) = \sum_{H \in \mathcal{S}_T} \delta^{|H|} \Big(q + \beta(2\alpha^H - 1) \Big).$$

Using the representation

$$2\alpha^H - 1 = A_H + B_H(2\alpha - 1),$$

we can collect the terms that depend on α and write

$$\Delta_T(\alpha; \delta) = q \, N_T(\delta) + \beta K_T(\delta) + \beta M_T(\delta)(2\alpha - 1).$$

Hence the cutoff is obtained by solving $\Delta_T(\hat{\alpha}_T; \delta) = 0$, which gives

$$\hat{\alpha}_T(\delta) = \frac{1}{2} - \frac{q}{2\beta} \frac{N_T(\delta)}{M_T(\delta)} - \frac{1}{2} \frac{K_T(\delta)}{M_T(\delta)}.$$

Differentiating with respect to δ,

$$\frac{\partial \hat{\alpha}_T}{\partial \delta} = \frac{q_E - q_I}{2\beta} \Big(\frac{N_T}{M_T} \Big)' - \frac{1}{2} \Big(\frac{K_T}{M_T} \Big)'.$$

It remains to rewrite the two derivatives on the right-hand side in a more useful form.

Set

$$\omega_H(\delta) := \frac{\delta^{|H|} B_H}{M_T(\delta)}, \qquad \lambda_H := \frac{A_H}{B_H}.$$

Then $\sum_{H \in \mathcal{S}_T} \omega_H(\delta) = 1$, and

$$\frac{N_T}{M_T} = \sum_{H \in \mathcal{S}_T} \omega_H \frac{1}{B_H}, \qquad \frac{K_T}{M_T} = \sum_{H \in \mathcal{S}_T} \omega_H \lambda_H.$$

For any collection $\{g_H\}_{H \in \mathcal{S}_T}$ that does not depend on δ, a direct differentiation shows that

$$\delta \frac{d}{d\delta} \sum_{H \in \mathcal{S}_T} \omega_H g_H = \mathrm{Cov}_\omega(|H|, g_H).$$

Applying this identity with $g_H = 1/B_H$ and $g_H = \lambda_H$, we obtain

$$\delta\Big(\frac{N_T}{M_T}\Big)' = \mathrm{Cov}_\omega\Big(|H|, \frac{1}{B_H}\Big), \qquad \delta\Big(\frac{K_T}{M_T}\Big)' = \mathrm{Cov}_\omega(|H|, \lambda_H).$$

Substituting these expressions into the derivative formula above yields

$$2\delta \frac{\partial \hat{\alpha}_T}{\partial \delta} = \frac{q_E - q_I}{\beta} \mathrm{Cov}_\omega\Big(|H|, \frac{1}{B_H}\Big) - \mathrm{Cov}_\omega(|H|, \lambda_H).$$

The conclusions now follow immediately from this identity. If

$$\mathrm{Cov}_\omega\Big(|H|, \frac{1}{B_H}\Big) \geq 0 \quad \text{and} \quad \mathrm{Cov}_\omega(|H|, \lambda_H) \leq 0,$$

then the right-hand side is nonnegative, so $\hat{\alpha}_T$ is increasing in δ. If instead

$$\mathrm{Cov}_\omega\Big(|H|, \frac{1}{B_H}\Big) > 0,$$

then the same identity can be rearranged as

$$\frac{\partial \hat{\alpha}_T}{\partial \delta} > 0 \quad \Longleftrightarrow \quad q_E - q_I > \beta \frac{\mathrm{Cov}_\omega(|H|, \lambda_H)}{\mathrm{Cov}_\omega\Big(|H|, \frac{1}{B_H}\Big)} = \phi_T(\delta),$$

Since this holds for T, it also holds for the limiting case. □

Proof for Lemma 2 We want to show that MR_i is weakly increasing in ℓ_i for both platforms.

Consider platform I. Since $\alpha_0 = \hat{\alpha} + \ell_I$, for any history $H = (x_1, \ldots, x_t)$ of length t,

$$\alpha^H = \rho^t \alpha_0 + (1 - \rho) \sum_{s=1}^{t} \rho^{t-s} x_s.$$

Hence α^H is increasing in α_0.

Since the absorbing conditions remain unchanged, the sets of histories that first reach absorption are

$$\mathcal{H}^I = \left\{ H : \alpha^H > \frac{\hat{\alpha}}{\rho} \right\}, \qquad \mathcal{H}^E = \left\{ H : \alpha^H < 1 - \frac{1 - \hat{\alpha}}{\rho} \right\}.$$

Therefore, as ℓ_I increases, $\mathcal{H}^I$ weakly expands, while $\mathcal{H}^E$ weakly shrinks.

The value function admits the pruned path-sum representation

$$V_I(\alpha_0) = \sum_{H \in \mathcal{H}} \delta^{|H|} \Delta(\alpha^H) + \sum_{H \in \mathcal{H}^I} \delta^{|H|} T_I(\alpha^H) + \sum_{H \in \mathcal{H}^E} \delta^{|H|} T_E(\alpha^H).$$

At the same level of α_0, when $\mathcal{H}^I$ expands and $\mathcal{H}^E$ shrinks, $V_I(\alpha_0)$ increases.

Because the increase in α_0 changes absorption sets, $V_I(\alpha_0)$ is piecewise affine in α_0, with kinks at the points where the absorbing sets change. Let the breakpoints be

$$t_k \in (\hat{\alpha}, 1), \qquad k \geq 1,$$

and write, on each open interval (t_{k-1}, t_k),

$$V_I(\alpha_0) = a_k + b_k \alpha_0.$$

At each kink t_k, the value function is continuous:

$$V_I(t_k^-) = V_I(t_k^+).$$

Therefore,

$$a_k + b_k t_k = a_{k+1} + b_{k+1} t_k,$$

so that

$$a_{k+1} = a_k + t_k(b_k - b_{k+1}).$$

Now take any $\alpha \in (t_k, t_{k+1})$. Then

$$V_I(\alpha) = a_{k+1} + b_{k+1}\alpha.$$

If moving from a point below t_k to α did not change the absorption pattern, then the value at α would be

$$a_k + b_k \alpha.$$

But once the change in absorption is taken into account that the set $\mathcal{H}^I$ is larger and the set $\mathcal{H}^E$ is smaller at α, then

$$V_I(\alpha) > a_k + b_k \alpha.$$

Using the continuity condition,

$$V_I(\alpha) - (a_k + b_k\alpha) = (a_{k+1} + b_{k+1}\alpha) - (a_k + b_k\alpha)$$

$$= (b_{k+1} - b_k)(\alpha - t_k).$$

Since $\alpha > t_k$ and

$$V_I(\alpha) - (a_k + b_k\alpha) > 0,$$

it follows that

$$b_{k+1} - b_k > 0.$$

Thus the slope b_k is increasing in k, so $V_I(\alpha_0)$ is convex in α_0. Therefore $MR_I(\ell_I)$ is weakly increasing.

The proof for E is symmetric, so $MR_E(\ell_E)$ is also weakly increasing. □

Proof for Proposition 6

We have already established the equilibrium existence condition in the paper. It remains to show that, holding the rival's marketing level fixed at 0, and suppose $\hat{\alpha} + \ell_I = 1 - \hat{\alpha} + \ell_E$,

$$AR(\ell_I) < AR(\ell_E).$$

This is equivalent to showing that

$$\frac{V_I(\ell_I)}{\ell_I} < \frac{V_E(\ell_E)}{\ell_E}.$$

First, for a given common post-marketing focality level, because $q_E > q_I$, we have

$$V_I(\ell_I) < V_E(\ell_E).$$

To prove that the average return of platform E with ℓ_E is larger than I with ℓ_I, it remains to show that ℓ_E has a stronger effect on absorption than ℓ_I.

To make this comparison precise, we distinguish the focality processes of the two platforms. Let

$$\alpha_I = \alpha_0 + \ell_I$$

denote platform I's initial focality, and let α_I^H be platform I's focality after history H. Similarly, let

$$\alpha_E = 1 - \alpha_0 + \ell_E$$

denote platform E's initial focality, and let α_E^H be platform E's focality after history H.

Consider a pair of complementary histories (H_I, H_E), that they are not absorbing.

For platform I, after ℓ_I, the histories are absorbing if the following conditions hold.

$$\alpha_I^{H_I} > \frac{\hat{\alpha}}{\rho}, \quad \text{I-absorbing}$$

$$\alpha_I^{H_E} < 1 - \frac{1-\hat{\alpha}}{\rho}, \quad \text{E-absorbing}$$

From platform E's side, the histories are absorbing if:

$$\alpha_E^{H_E} > \frac{1-\hat{\alpha}}{\rho}, \quad \text{E-absorbing}$$

$$\alpha_E^{H_I} < 1 - \frac{\hat{\alpha}}{\rho}, \quad \text{I-absorbing.}$$

Since

$$\frac{\hat{\alpha}}{\rho} > \frac{1-\hat{\alpha}}{\rho},$$

we have:

$$\alpha_I^{H_I} > \frac{\hat{\alpha}}{\rho} \quad \Rightarrow \quad \alpha_E^{H_E} > \frac{1-\hat{\alpha}}{\rho},$$

while the opposite does not hold. Thus, it is easier for $\alpha_E^{H_E}$ to reach E-absorbing, than $\alpha_I^{H_I}$ reaches I-absorbing. On the other hand, since

$$1 - \frac{1-\hat{\alpha}}{\rho} > 1 - \frac{\hat{\alpha}}{\rho},$$

we have:

$$\alpha_E^{H_I} < 1 - \frac{\hat{\alpha}}{\rho} \quad \Rightarrow \quad \alpha_I^{H_E} < \frac{1-\hat{\alpha}}{\rho},$$

while the opposite does not hold. Thus, it is easier for $\alpha_I^{H_E}$ to reach E-absorbing, than $\alpha_E^{H_I}$ to reach I-absorbing. Thus, ℓ_E changes the absorption in favor of E more than ℓ_I, and the average return of E is higher. □

Proof of lemma 3

We want to show that: $MR_I(\ell_I)$ and $MR_E(\ell_E)$ are weakly increasing, and for $\ell_I = \ell_E$,

$$MR_I(\ell_I) \leq MR_E(\ell_E).$$

Denote:

$$V_i(\ell_i) = V_i(\alpha_0 \mid \rho_i = \rho + \ell_i, \; \rho_j = \rho - \ell_i).$$

First, we prove for platform I.

Recall that:

$$V_I(\alpha_0) = \sum_{H \in \mathcal{H}} \delta^{|H|}(q + \beta(2\alpha^H - 1)) + \sum_{H \in \mathcal{H}^I} \delta^{|H|} T_I(\alpha^H) + \sum_{H \in \mathcal{H}^E} \delta^{|H|} T_E(\alpha^H).$$

For each pruning rule $\mathcal{P}$ that determines the sets $\mathcal{H}$, $\mathcal{H}^I$, $\mathcal{H}^E$, write the value of platform I as $V_I^{\mathcal{P}}(\ell)$. We want to show:

$$\ell_2 > \ell_1 \qquad \Longrightarrow \qquad \frac{\partial V_I(\ell_2)}{\partial \ell_2} \geq \frac{\partial V_I(\ell_1)}{\partial \ell_1}.$$

We have for every history H, $\alpha^H(\ell_I)$ is weakly increasing in ℓ_I.

Because the absorbing thresholds change only when a boundary is crossed, the pruning rule is piecewise constant in ℓ_I. Hence there exist cutoffs

$$\bar{\ell}_0 < \bar{\ell}_1 < \cdots < \bar{\ell}_M$$

such that on each open interval $(\bar{\ell}_{m-1}, \bar{\ell}_m)$, the pruning rule is fixed, say equal to $\mathcal{P}_m$.

Fix such an interval. Since the absorption sets are fixed, term-by-term differentiation gives

$$\frac{d}{d\ell_I} V_I^{\mathcal{P}_m}(\ell_I) = \sum_{H \in \mathcal{H}^{\mathcal{P}_m}} \delta^{|H|} \cdot 2\beta \cdot \frac{\partial \alpha^H(\ell_I)}{\partial \ell_I}$$
$$+ \sum_{H \in \mathcal{H}_I^{\mathcal{P}_m}} \delta^{|H|} T_I'(\alpha^H(\ell_I)) \frac{\partial \alpha^H(\ell)}{\partial \ell_I} + \sum_{H \in \mathcal{H}_E^{\mathcal{P}_m}} \delta^{|H|} T_E'(\alpha^H(\ell_E)) \frac{\partial \alpha^H(\ell)}{\partial \ell_I},$$

which is positive. Thus the only issue is what happens at a switching point $\bar{\ell}_m$.

Let $D^- V_I(\bar{\ell}_m)$ and $D^+ V_I(\bar{\ell}_m)$ denote the left and right derivatives. Histories whose status does not change at $\bar{\ell}_m$ contribute the same term to both one-sided derivatives, so they cancel in the difference

$$D^+ V_I(\bar{\ell}_m) - D^- V_I(\bar{\ell}_m).$$

Hence we only need to consider histories that switch status when the pruning rule changes.

Each such history can only move in one of the following three directions:

(a) from $\mathcal{H}$ to $\mathcal{H}_I$. For such a history H, the change in its contribution to the derivative is

$$\delta^{|H|} \Big[T_I'(\alpha^H(\bar{\ell}_m)) - 2\beta \Big] \frac{\partial \alpha^H(\bar{\ell}_m)}{\partial \ell} \geq 0$$

by assumption (i) and Step 1.

(b) from $\mathcal{H}_E$ to $\mathcal{H}$. The change in derivative is

$$\delta^{|H|} \Big[2\beta - T_E'(\alpha^H(\bar{\ell}_m)) \Big] \frac{\partial \alpha^H(\bar{\ell}_m)}{\partial \ell} \geq 0.$$

(c) from $\mathcal{H}_E$ to $\mathcal{H}_I$. The change in derivative is

$$\delta^{|H|} \Big[T_I'(\alpha^H(\bar{\ell}_m)) - T_E'(\alpha^H(\bar{\ell}_m)) \Big] \frac{\partial \alpha^H(\bar{\ell}_m)}{\partial \ell} \geq 0.$$

Therefore every switching history contributes weakly positively to $D^+V_I(\bar{\ell}_m) - D^-V_I(\bar{\ell}_m)$, so

$$D^+V_I(\bar{\ell}_m) \geq D^-V_I(\bar{\ell}_m).$$

Thus the derivative is weakly increasing within each region and can only jump upward when the pruning rule changes. It follows that the one-sided derivative of $V_I(\ell)$ is weakly increasing on the whole domain. Hence $V_I(\ell_I)$ has weakly increasing marginal returns in ℓ_I. Same proof applies for platform E.

It remains to show that $MR_I(\ell) < MR_E(\ell)$. Since the increasing rate on value functions depends on the change in absorptions, we show that at ℓ, it changes pruning rule more for platform E than platform I.

To make this comparison precise, we distinguish the focality processes of the two platforms. Let

$$\alpha_I = \hat{\alpha}$$

denote platform I's initial focality, and let α_I^H be platform I's focality after history H. Similarly, let

$$\alpha_E = 1 - \hat{\alpha}$$

denote platform E's initial focality, and let α_E^H be platform E's focality after history H. Consider a pair of complementary histories (H_I, H_E) that are not absorbing.

For platform I, after $\ell_I = \ell$, the histories are absorbing if the following conditions hold.

$$\alpha_I^{H_I} > \frac{\hat{\alpha}}{\rho + \ell}, \quad \text{I-absorbing}$$

$$\alpha_I^{H_E} < 1 - \frac{1 - \hat{\alpha}}{\rho - \ell}, \quad \text{E-absorbing}$$

From platform E's side, after $\ell_E = \ell$, the histories are absorbing if:

$$\alpha_E^{H_E} > \frac{1 - \hat{\alpha}}{\rho + \ell}, \quad \text{E-absorbing}$$

$$\alpha_E^{H_I} < 1 - \frac{\hat{\alpha}}{\rho - \ell}, \quad \text{I-absorbing.}$$

Since

$$\alpha_i^{H_i} = \left(\prod_{s=0}^{t-1} \rho_s\right)\alpha_i + \sum_{n=1}^{t}\left[(1 - \rho_{t-n}) \prod_{s=t-n+1}^{t-1} \rho_s\right]x_{t-n} = \phi_1\alpha_i + \phi_2,$$

and with the same ordered sequence, ϕ_1 and ϕ_2 remain unchanged. When α_i changes from $\hat{\alpha}$ to $1 - \hat{\alpha}$, it makes the inequality harder to satisfy. Thus,

$$\alpha_I^{H_I} > \frac{\hat{\alpha}}{\rho + \ell} \quad \Rightarrow \quad \alpha_E^{H_E} < \frac{1 - \hat{\alpha}}{\rho + \ell},$$

while the opposite does not hold. It is easier for α^{H_E} to reach E-absorbing after E invests, than α^{H_I} reaches

I-absorbing after I invests. Similar for the other side, we have:

$$\alpha_E^{H_I} < 1 - \frac{\hat{\alpha}}{\rho - \ell} \quad \Rightarrow \quad \alpha_I^{H_E} < \frac{1 - \hat{\alpha}}{\rho + \ell},$$

while the opposite does not hold. Thus, it is easier for $\alpha_I^{H_E}$ to reach E-absorbing, than $\alpha_E^{H_I}$ to reach I-absorbing. Thus, ℓ_E changes the absorption in favor of E more than ℓ_I, and the average return of E is higher.

□

Proof for Proposition 7

The result follows immediately from the discussion in the paper and the previous lemma. □

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